THE MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

LONDON
G. BELL & SONS, LTD., PORTUGAL STREET, KINGSWAY, W.C. 2

Vol. XX., No. 239. JULY, 1936. 3s. Net.

CONTENTS.

Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 15s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette".

Change of Address should be notified to Miss Punnett, 17 Gower St., London, W.C. 1. If Copies of the "Gazette" fail for lack of such notification to reach a member, duplicate copies can be supplied only at the published price.

Subscriptions should be paid to the Hon. Treasurer, Mathematical Association, 29 Gordon Square, London, W.C. 1.

I

f

t

w

a
o

f
t
g
o
t
t

THE

MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

62 COLERAINE ROAD, BLACKHEATH, LONDON, S.E. 3

LONDON

G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY

VOL. XX. JULY, 1936. No. 239

SUMS OF POWERS OF RECIPROCALS.

BY A. LODGE.

FORMULAE are established below for

$$\Sigma_r(x) \equiv \Sigma_r \equiv \left(\frac{1}{x+1}\right)^r + \left(\frac{1}{x+2}\right)^r + \left(\frac{1}{x+3}\right)^r + \dots$$

for both odd and even integer values of r ($r > 1$), and $x \geqslant 0$; also for

$$\sigma\left(\frac{1}{x}\right) \equiv \left(1 - \frac{1}{x+1}\right) + \left(\frac{1}{2} - \frac{1}{x+2}\right) + \left(\frac{1}{3} - \frac{1}{x+3}\right) + \dots,$$

this series reducing to $1 + \frac{1}{2} + \frac{1}{3} + \dots + 1/x$, when x is an integer.

In 1920 I established the formula, a most useful one,

$$\sigma(1/x) = \gamma + \tfrac{1}{2} \log \{x(x+1)\} + 1/\{6x(x+1) + 1{\cdot}2\},$$

where γ is Euler's constant

$$0{\cdot}57721 \quad 56649 \quad 01532 \quad 86060 \quad 6,$$

and in 1929 gave * a simpler proof with extension to higher powers of $x(x+1)$ with formulae for Σ_{2n+1}, but not for Σ_{2n}.

My present object is to establish formulae for all the above series, for even as well as for odd values of r, the formulae being still in terms of various powers of x and $x+1$, but not of their product. One great advantage of this separation of x and $x+1$ is that, having found one series, the others are at once obtained by successive differentiation or integration. For this purpose we shall find that Σ_2 compels the first attention.

The initial formulae used are, ($x > 1$),

$$\frac{1}{x-1} - \frac{1}{x} = \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \dots, \quad \text{.........................(1)}$$

* *British Association Report*, 1929.

and $$\frac{1}{x}-\frac{1}{x+1}=\frac{1}{x^2}-\frac{1}{x^3}+\frac{1}{x^4}+\dots \quad \text{.......................(2)}$$

From (1) we get, by changing x into $x+1$ in successive lines,

$$\frac{1}{x}-\frac{1}{x+1}=\left(\frac{1}{x+1}\right)^2+\left(\frac{1}{x+1}\right)^3+\dots,$$

$$\frac{1}{x+1}-\frac{1}{x+2}=\left(\frac{1}{x+2}\right)^2+\left(\frac{1}{x+2}\right)^3+\dots,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

Thus, finally by addition,

$$1/x=\Sigma_2+\Sigma_3+\Sigma_4+\dots \quad \text{.......................(3)}$$

From (2), similarly,

$$\frac{1}{x+1}-\frac{1}{x+2}=\left(\frac{1}{x+1}\right)^2-\left(\frac{1}{x+1}\right)^3+\left(\frac{1}{x+1}\right)^4+\dots,$$

$$\frac{1}{x+2}-\frac{1}{x+3}=\left(\frac{1}{x+2}\right)^2-\left(\frac{1}{x+2}\right)^3+\left(\frac{1}{x+2}\right)^4+\dots,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

Thus $$\frac{1}{x+1}=\Sigma_2-\Sigma_3+\Sigma_4+\dots \quad \text{.......................(4)}$$

Formulae (3) and (4) provide useful tests of the accuracy of tables of the successive Σ functions when these are found.

We have now to find a formula for Σ_2: there are two such formulae, one based on (3), the other on (4), and both requiring symbolical use of the Bernoulli expansion

$$\frac{x}{e^x-1}=1-\frac{x}{2}+B_2\frac{x^2}{2!}-B_4\frac{x^4}{4!}+B_6\frac{x^6}{6!}-\dots,$$

where $B_2=\frac{1}{6},\quad B_4=\frac{1}{30},\quad B_6=\frac{1}{42},\quad B_8=\frac{1}{30},\quad \dots .$

It should be noted that if the sign of x is changed in the left-hand side, the only change on the right is in the sign of $\frac{1}{2}x$.

Now, taking (4) first, let y be such a function of x that

$$Dy=\Sigma_2,$$

then $$D^2y=-2!\Sigma_3,\quad D^3y=3!\Sigma_4,\quad \dots .$$

Thus $$y+\frac{1}{x+1}=y+Dy+D^2y/2!+D^3y/3!+\dots$$

$$=e^Dy,$$

or $$y=\frac{1}{e^D-1}\left(\frac{1}{x+1}\right),$$

and $$Dy=\frac{D}{e^D-1}\left(\frac{1}{x+1}\right).$$

Thus $\Sigma_2 = \dfrac{1}{x+1} + \dfrac{1}{2(x+1)^2} + \dfrac{1}{6(x+1)^3} - \dfrac{1}{30(x+1)^5} + \dots$ ....(5)

In the same way, from (3),

$$y - 1/x = y - Dy + D^2y/2! - D^3y/3! + \dots$$
$$= e^{-D}y,$$

so that
$$Dy = \frac{-D}{e^{-D}-1}\left(\frac{1}{x}\right),$$

whence
$$\Sigma_2 = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \dots \qquad \text{..............(6)}$$

The formula (5) and (6) express Σ_2 in series which can be used for calculation, but a better one can be obtained by their combination. Multiply (5) by $x+1$, and (6) by x, and subtract. Then

$$\Sigma_2 = \frac{1}{2}\left(x + \frac{1}{x+1}\right) - \frac{1}{6}\left\{\frac{1}{x^2} - \frac{1}{(x+1)^2}\right\} + \frac{1}{30}\left\{\frac{1}{x^4} - \frac{1}{(x+1)^4}\right\} - \dots,$$

or, using M_r to denote $x^{-r} + (x+1)^{-r}$,

and N_r to denote $x^{-r} - (x+1)^{-r}$,

$$\Sigma_2 = \tfrac{1}{2}M_1 - \tfrac{1}{6}N_2 + \tfrac{1}{30}N_4 - \tfrac{1}{42}N_6 + \tfrac{1}{30}N_8 - \tfrac{5}{66}N_{10} + \dots \qquad \text{......(7)}$$

This is the most convergent formula I can find.

By differentiation, we get

$$2!\,\Sigma_3 = \tfrac{1}{2}M_2 - \tfrac{2}{6}N_3 + \tfrac{4}{30}N_5 - \tfrac{6}{42}N_7 + \tfrac{8}{30}N_9 - \tfrac{50}{66}N_{11} + \dots .$$
$$3!\,\Sigma_4 = M_3 - N_4 + \tfrac{2}{3}N_6 - N_8 + \tfrac{24}{10}N_{10} - \tfrac{25}{3}N_{12} + \dots .$$
$$4!\,\Sigma_5 = 3M_4 - 4N_5 + 4N_7 - 8N_9 + 24N_{11} - \dots,$$

where the run of the numerical coefficients is not so good, but the N values are smaller.

I have in this way calculated the several functions to 20 decimals between $x = 58{\cdot}0$ and $x = 58{\cdot}9$, and extended the tables above and below these values of x by using the obvious identities

$$\Sigma_r(x+1) = \Sigma_r(x) - (x+1)^{-r},$$
$$\Sigma_r(x-1) = \Sigma_r(x) + x^{-r}.$$

By integration, and adjustment of constants, we get

$$\left(1 - \frac{1}{x+1}\right) + \left(\frac{1}{2} - \frac{1}{x+2}\right) + \left(\frac{1}{3} - \frac{1}{x+3}\right) + \dots$$
$$= \frac{1}{2}\{\log x + \log(x+1)\} + \frac{1}{6}\left(\frac{1}{x} - \frac{1}{x+1}\right) - \frac{1}{90}\left\{\frac{1}{x^3} - \frac{1}{(x+1)^3}\right\} + \dots$$
$$= \tfrac{1}{2}\{\log x + \log(x+1)\} + \tfrac{1}{6}N_1 - \tfrac{1}{90}N_3 + \tfrac{1}{210}N_5 - \tfrac{1}{210}N_7 + \tfrac{5}{594}N_9 - \dots .$$

These series are all derivatives (positive or negative) of $\log \Gamma(1+x)$.

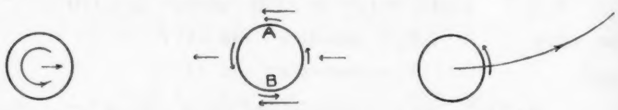
A. L.

THE PHYSICS OF SPORT.*

By Sir G. T. Walker.

If anything were calculated to make me anxious to do justice to my theme to-night it would be the association with your society of the men to whom I owe my earliest introduction to dynamics—at St. Paul's School to Mr. Pendlebury, your Secretary, and at Trinity College, Cambridge, to Professor Forsyth, your incoming President. The interest that they implanted has survived for half a century; and the applications to sport that I propose to describe are the immediate outcome of that interest.

The dynamical effect which is most widely known in ball games and is perfectly familiar to all of you is the curling of the path of a rotating sphere when moving through the air. It is seen in the slice of a golf ball, the cut on a tennis ball and the swerve in cricket; and the explanation, often attributed to Tait, had been previously given by Rayleigh as well as, in general terms, by Newton (I).



(*a*) Ball moving left to right through stationary air.

(*b*) Centre of ball treated as at rest and air moving right to left. The speed of the air much greater at A than at B, so pressure greater at B than at A.

(*c*) The resulting path.

Fig. I.

But a racket court provides another anomaly; for there the ball is heavily undercut by the server and yet rebounds in the downward

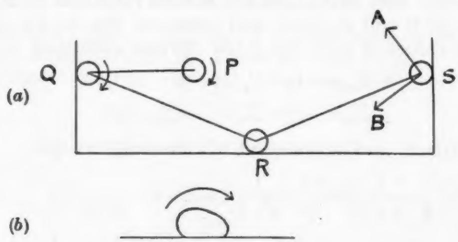


Fig. II.

direction off the back wall (II*a*) instead of upwards as we should expect: obviously the direction of rotation must be reversed on

* A paper at the Annual Meeting of the Mathematical Association, January 2nd, 1936.

impact on the front wall. Now one of the most conspicuous features in the court is the large size of the marks left by the ball flattening out on the walls: so that the idea of taking moments about the point of impact of a rigid sphere and thereby retaining the direction of rotation is impossible. The more natural supposition is that the ball flattens out to the moment of greatest compression (II*b*) and then rebounds with the velocities of all its parts reversed, bearing a constant ratio, say *e*, to their values on impact: and this idea, which gives a reversed spin of *e* times the original spin, explains our paradox.

Oddly enough the curve in the path of a curling stone has the same relation to its direction of spin as that of a ball moving through the air. For its production it is essential that the stone shall be in contact with the ice, not at the central point, but over a circular ring about 6″ in diameter (III). If we call R_1 and R_2 the vertical thrusts

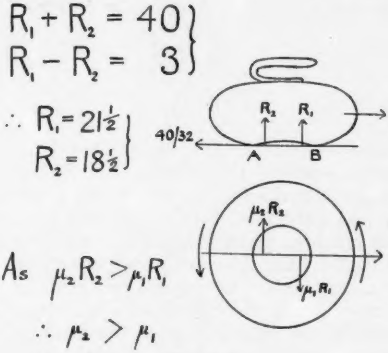


Fig. III.

over the front and back half of the circle it is clear that, in view of the friction of the ice, R_1 must be greater than R_2. Suppose that μ_1 and μ_2 are the coefficients of friction on the front and back half of the stone, if unequal; then as a first approximation the resultants of the frictions on the two halves will be sideways and proportional to $\mu_1 R_1$ and $\mu_2 R_2$. If the stone were sliding over a glass plate, μR_1 would be greater than μR_2, and in such a case the revolving object does actually curve to the right. Since a stone on ice curves to the left, $\mu_2 R_2 > \mu_1 R_1$, and as $R_1 > R_2$, $\mu_2 > \mu_1$. Now we know that if the ice is not far from the melting point pressure in excess of a critical value melts the ice and reduces the friction: it is this property that makes skating possible, for it produces a film of water between the skate and the ice. So we are not surprised that $\mu_2 > \mu_1$; and if our explanation is right a stone should refuse to curl when its pressure cannot melt the ice, *i.e.* when it is very cold. This agrees with experience.

Curiously enough there is yet another case in which rotation sets up deviation in the same direction as for a ball. It is that of a falling long rectangular strip of cardboard. As Maxwell pointed out, if it starts spinning as shown in the figure (IV*a*) the velocity will be least

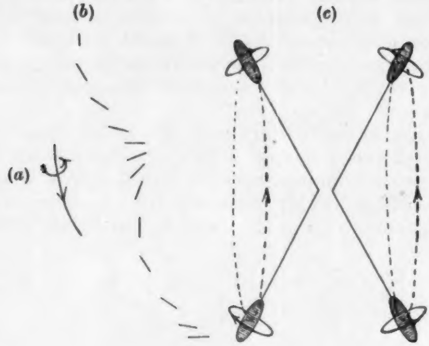


FIG. IV.

after the plane has passed through the horizontal, and greatest after passing through the vertical (IV*b*). So the couples tending to increase the spin will be greater than those tending to decrease it ; and the spin will grow. Also the horizontal forces will be greater after the vertical position has been passed through ; and the resulting deviation will be that indicated. An example of this effect may be seen in the bull-roarer, an implement used in magical rites by primitive man over a large part of the earth. As you will observe (IV*c*) the string describes a cone on the side indicated by Maxwell's theory, and when it has twisted so far that it must start untwisting, the cone shifts over to the other side.

Some effects of the 'nap' of a cloth may now be considered. A billiard player learns by experience that the path of a ball travelling slowly with much "side", that is, with a rapid spin about a vertical axis, will be diverted to the right or left according as the spinning motion on the right side is with or against the nap (V*a*). The deflection may be something like an inch in the length of the table. I have seen pages of mathematical analysis vainly devoted to the subject. But if we state the result in the form that the path bends away from the side on which the nap is being rubbed up and towards that on which it is being stroked down the explanation is obvious : for on the rubbed-up side the effective surface is higher than on that stroked down and the ball moves, as it were, on an inclined plane. This agrees with the experience that there is no deviation whatever on a ball travelling with the nap unless the spin is fast and the forward motion slow ; otherwise appreciable rubbing up will not occur.

A curious result is seen on rolling a coin on a cloth with a nap.

The fundamental fact is that if the coin is rolled in the direction at right angles to the nap (V*b*) its path bends in the direction opposite to that to which the nap points. Two less conspicuous results follow.

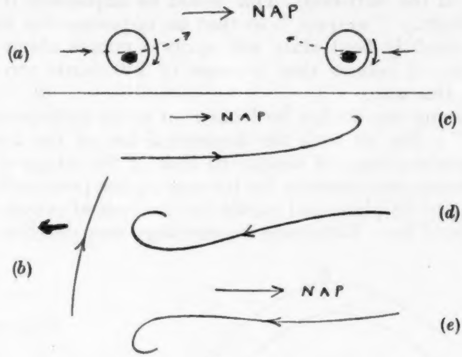


FIG. V.

If the coin is rolled along the nap (V*c*) its path curls more and more to one side or the other ; but when rolled against the nap there is always a reversal of the curvature (V*d*, *e*).*

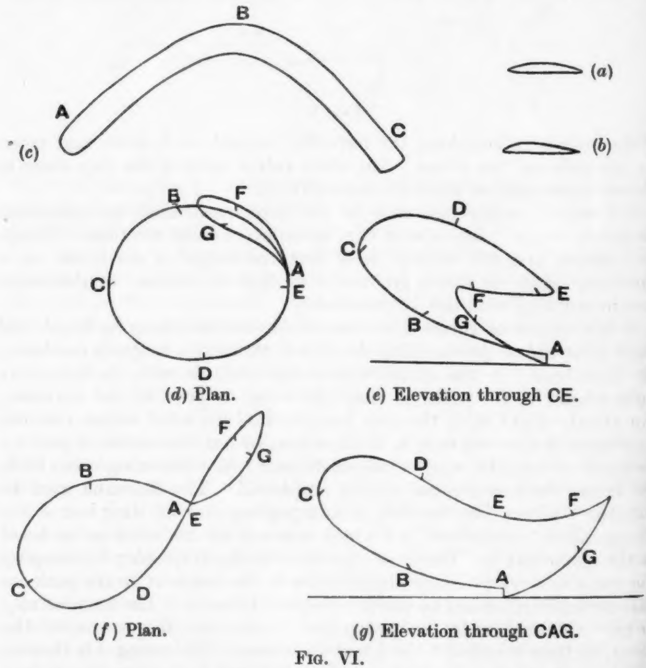
We were looking just now at an implement used by primitive people in magic ; let us now turn to some of their weapons. Slings and stones are still widely used, but the range of an arrow or a throwing spear is much greater than that of a stone of the same weight starting with the same velocity.

A few efforts to throw a wooden stick five feet long by hand will show that unless precautions are taken there is a marked tendency for it to travel in the stable transverse attitude with its length at right angles to its path, so that the range is poor in the extreme. For steady flight with the axis longitudinal we must either provide resistance in the rear end, as in an arrow, or put the centre of gravity forward, or spin the weapon about its axis : in a throwing spear both the latter devices are commonly employed. The Romans used to spin the " pilum " by treating it as a peg-top and winding round it a thong called " amentum ", of which one end was retained in the hand as the spear left it. The spin also adds to the trajectory by keeping the spear always pointing slightly above the tangent to its path, so that its fall is retarded by the air pressure below it. The natural way to give the spin is by wrapping the thumb and fingers round the spear, so that it rolls off the hand on release. The assegai is thrown in this way and flies like an arrow from a bow ; its penetration is very great. But if the spear is not too heavy a separate implement, a throwing stick like the Australian " womera," may be used with

* It may be noted that in velveteen the pile is at right angles to the cloth ; but in velvet the pile is inclined in one direction, *i.e.* there is a " nap ". Experiments with a coin may fail unless the cloth has been brushed in the direction of the nap.

advantage. This enables an invaluable flick of the wrist to be employed, and, strangely enough, can impart a considerable spin although the impetus is all given by a smooth peg in contact with the centre of the butt-end. This would be impossible if the spear were not slightly " whippy ", so that an imperceptible flick to one side in its slightly bent state will apply a couple about its longitudinal axis. I believe that a range of 150 yards can easily be attained in this way.

A boomerang can to-day be looked on as an anticipation of the " autogiro " ; for, as with the horizontal fan of the autogiro, its rotation provides support similar to that of the wings of an aeroplane. The couples necessary for its steering are produced partly by the warping of its plane and partly by the lack of symmetry in its cross-section (VI*a*). Returning boomerangs may describe paths of



(*d*) Plan. (*e*) Elevation through CE.

(*f*) Plan. (*g*) Elevation through CAG.

Fig. VI.

various types: one with two loops in front is shown in (*d*), (*e*), and a figure of eight in (*f*), (*g*). But for an ordinary missile to be thrown at an animal the shape is designed to give a straight and very flat trajectory. Its efficiency is considerable ; when my range with a cricket ball weighing $4\frac{3}{4}$ oz. was seventy yards I could throw

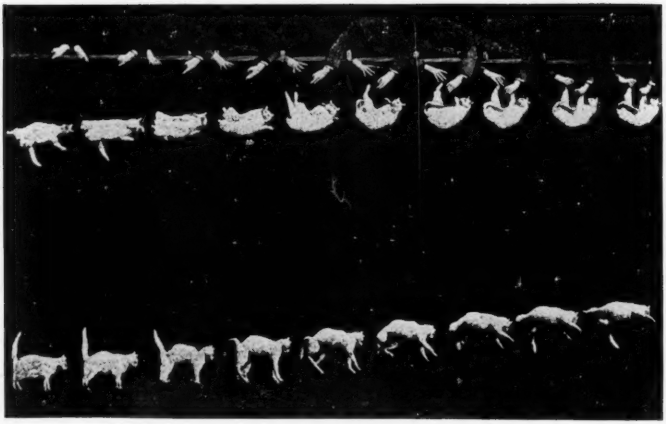


Fig. VII.

THE PHYSICS OF SPORT.

a straight-going boomerang, weighing twice as much, a distance of 185 yards. One of my literary friends used to maintain that a boomerang alighted rotating faster than when it left the thrower's hand ; and I used to reply, with the cocksureness of a mathematician, that such a thing was impossible. But we are now familiar with the way in which, except in recent patterns, the rotation of the horizontal fan of an autogiro was provided, not by the direct drive of an engine, but by the forward rush of the machine through the air. So my literary friend may have been partly right after all.*

A well-known tool of the stone age is the hatchet head or " celt ", still in widespread use : and under favourable conditions this has curious properties. When placed on a fixed plate of clean glass many of them will spin in one direction and not in the other. The celt is oblong and at the point of contact with the plate the lines of curvature are not, in general, precisely parallel with the dynamical axes. There is rotational asymmetry, and this shows itself when the celt is spun. The theory † brought out another paradox : when tapped at one end rotation is set up, and the direction of this may be reversed merely by raising the centre of gravity. ‡

My last example shall be taken from living beings. When doing a high jump of six feet a man approaches the bar in a vertical attitude, and when he clears it he is nearly horizontal ; so, unless precautions are taken, he will during his descent, for a second time, turn through nearly two right angles ; and his alighting would be most unpleasant but for violent contortions executed on the way down, which take some time to learn. Perhaps the best exponent of this art is the cat, who, if suspended by the paws with her back only a few inches above a table, and released, will fall on her feet (VII). I believe that the cat at once holds out its long hind-paws at right angles to its body and draws in its fore-paws, so that by a rapid contortion which rotates its forward portion by about 180° the hind-part, with its bigger moment of inertia, will rotate through only say 60° in the opposite direction ; the contortion is then reversed with the fore-paws extended and the hind-paws withdrawn. They are completely spun round and catch the table, so that the forward part of the body can be quickly spun round and the cat is on its feet. ¶ I will illustrate this by getting on to a platform mounted so as to rotate without appreciable friction and, by repeated use of my arms, turn my body completely round as often as I wish, although at no instant is there the slightest angular momentum about the vertical. G. T. W.

* Something may be learned about boomerangs by the flicking of small models cut with scissors from a visiting card.

† *Q. J. Pure and Applied Mathematics*, 1896. A diagram, not given in the article in the *Q. J.*, and a description is in *Journal of Scientific Instruments* VIII, No. 2, February 1931.

‡ It is vital that the glass plate shall rest flat on a firm table ; otherwise the energy of rotation of the celt is wasted in shaking the table.

¶ A glance at a cat licking its back will show that the amount of twisting of the spinal column postulated is by no means difficult. But it performs the whole operation of rotation in the air in about an eighth of a second !

NUMERICAL EQUATIONS WITH COMPLEX COEFFICIENTS.

By E. H. Neville.

In response to Dr. Lidstone's letter in the May number of the *Gazette*, I have solved the equation

$$z^2-(2+i)z+(4+3i)=0,$$

a quadratic equation with complex coefficients chosen at random, by a process, or rather a succession of processes, of complete generality.

1. Let the roots be c, d, or in terms of modulus and angle, C cis γ, D cis δ. Since one root is not the conjugate of another, it will be a coincidence if the moduli are equal, and we may expect * the roots to be separated by the root-squaring process. The equation whose roots are c^{16}, d^{16} is

$$z^2-p_4z+q_4=0,$$

where

$$p_4=(-1{\cdot}36000+0{\cdot}02946i)\times 10^6=1{\cdot}3603\times 10^6 \text{ cis } 178^\circ\ 46',$$

$$q_4=(-0{\cdot}98248-1{\cdot}16749i)\times 10^{11}=1{\cdot}5259\times 10^{11} \text{ cis } 229^\circ\ 55',$$

and the equation whose roots are c^{32}, d^{32} is

$$z^2-p_5z+q_5=0,$$

where $$p_5=p_4^{\,2}-2q_4, \quad q_5=q_4^{\,2}.$$

The ratio of $|p_5|$ to $|q_5/p_5|$ is of the order of 10^2, and therefore p_5 and q_5/p_5 are accurate to about one part in 10^2 as approximations to c^{32} and d^{32}. As approximations to c^{16} and d^{16} we should expect p_4 and q_4/p_4 to be crude, but it is instructive to compare two lines of calculation, one in which the first estimates are p_4 and q_4/p_4, the other in which the first estimates are

$$p_4-(q_4/p_4) \quad \text{and} \quad q_4/\{p_4-(q_4/p_4)\}.$$

We have

$$q_4/p_4=1{\cdot}1217\times 10^5 \text{ cis } 51^\circ\ 9'=(7{\cdot}0362+8{\cdot}7357i)\times 10^4,$$

$$p_4-(q_4/p_4)=(-1{\cdot}43036-0{\cdot}05790i)\times 10^6=1{\cdot}4315\times 10^6 \text{ cis } 182^\circ\ 19',$$

$$q_4/\{p_4-(q_4/p_4)\}=1{\cdot}0659\times 10^5 \text{ cis } 47^\circ\ 36'=(7{\cdot}1876+7{\cdot}3460i)\times 10^4,$$

and therefore, according to the approximation adopted,

$$C=2{\cdot}4174, \quad \gamma=11^\circ\ 10'+m\times 22^\circ\ 30',$$
$$D=2{\cdot}0683, \quad \delta=3^\circ\ 12'+n\times 22^\circ\ 30',$$

or

$$C=2{\cdot}4252, \quad \gamma=11^\circ\ 24'+m\times 22^\circ\ 30',$$
$$D=2{\cdot}0617, \quad \delta=2^\circ\ 58'+n\times 22^\circ\ 30',$$

where m, n are integers.

* It was an odd oversight to think that squaring is less effective with complex coefficients than with real, since the contrary is the case; in general, if any coefficient is complex the roots are completely isolated sooner or later.

2. For the next step, the determination of m and n, I have no method which does not involve an amount of labour that seems disproportionate. It would be possible to substitute the available values of $C \operatorname{cis} \gamma$ and $D \operatorname{cis} \delta$ in turn in the original equation, but the number of trials that may be necessary is doubled with each rise in the degree of the Gräffe equation, and it is hard to know in advance how close a fit is to be taken as evidence that the correct choice is being made. The principle of the direct process which is recommended is ascribed by Carvallo to Goursat.

If $t = \cot \frac{1}{2}\theta$, then $r \operatorname{cis} \theta = r(t+i)/(t-i)$; conversely, if

$$z = r(t+i)/(t-i)$$

where $r = |z|$, then t is real and $2 \operatorname{arc} \cot t$ is the angle of z. It follows that if the equation $f(z) = 0$ in z has one and only one root with modulus r, the equation $f\{r(t+i)/(t-i)\} = 0$ in t has one and only one real root ; t being supposed real, this equation in t is equivalent to two equations with real coefficients, and these equations have one and only one common root. We can replace two equations

$$u(t) \equiv u_0 t^n + u_1 t^{n-1} + \dots + u_n = 0, \quad v(t) \equiv v_0 t^n + v_1 t^{n-1} + \dots + v_n = 0$$

of degree n by the two equations

$$v_0 u(t) - u_0 v(t) = 0, \quad \{v_n u(t) - u_n v(t)\}/t = 0$$

of degree $n-1$, these again by two equations of lower degree, and so on, until finally we have two equations whose equivalence, within the limits of approximation employed, is a verification that the value of the modulus is admissible ; these ultimate equations are usually linear, but in any case have only one real root.

As we are using this process not for the close evaluation of the angles but only for the identification of m and n, any attempt to maintain accuracy even in the second decimal figure would be waste of effort.

Corresponding to $|z| = 2{\cdot}42$, we have for t the two quadratic equations

$$5{\cdot}02\,t^2 + 6{\cdot}00\,t - 14{\cdot}70 = 0, \quad 0{\cdot}58\,t^2 + 3{\cdot}72\,t - 5{\cdot}42 = 0,$$

implying the two linear equations

$$15{\cdot}19\,t - 18{\cdot}78 = 0, \quad 18{\cdot}78\,t - 22{\cdot}16 = 0,$$

of which the roots are $\cot 39°$ and $\cot 40°$; the angle γ is therefore about $80°$, and m is 3.

With $|z| = 2{\cdot}06$, the quadratic equations are

$$4{\cdot}12\,t^2 + 6{\cdot}00\,t - 12{\cdot}36 = 0, \quad 0{\cdot}94\,t^2 + 0{\cdot}48\,t - 5{\cdot}06 = 0,$$

the linear equations are

$$3{\cdot}62\,t + 9{\cdot}73 = 0, \quad 9{\cdot}73\,t + 24{\cdot}43,$$

the roots of the linear equations are $\cot 160°$ and $\cot 158°$, the angle δ is about $320°$, and n is identified as 14.

3. We have now two approximations to the roots c, d:

$$\text{(i)}\quad c = 2{\cdot}417 \text{ cis } 78^\circ\, 40' = 0{\cdot}475 + 2{\cdot}370i,$$
$$d = 2{\cdot}068 \text{ cis } 318^\circ\, 12' = 1{\cdot}542 - 1{\cdot}378i;$$

$$\text{(ii)}\quad c = 2{\cdot}4252 \text{ cis } 78^\circ\, 54' = 0{\cdot}4668 + 2{\cdot}3799i,$$
$$d = 2{\cdot}0617 \text{ cis } 317^\circ\, 58' = 1{\cdot}5313 - 1{\cdot}3804i.$$

Two questions remain. Do we know the accuracy of these approximations? How are we to improve upon them?

It is a simple matter to assign some limit to the error of a Gräffe approximation, but any such limit allows for the most unfavourable combination of circumstances, and the approximation in an actual case is usually much better than can be guaranteed. Here for example we see from (ii) that the values in (i) are in error by only about 1 in 100, although the fraction $|q_4/p_4^2|$ which we are neglecting is about 1/12. To estimate the error in (i) by completing the calculation of (ii) and still to remain without any estimate of the error in (ii) is certainly a wasteful process. There is an alternative, the iterative improvement of any known approximation.

In terms of the real numbers x, y, the equation in z is equivalent to the pair of equations

$$f(x, y) \equiv x^2 - y^2 - 2x + y + 4 = 0, \quad g(x, y) \equiv 2xy - x - 2y + 3 = 0.$$

Substituting $a+h$, $b+k$ for x, y and neglecting all but the linear terms in h, k, we have h, k determined by the pair of equations

$$f(a, b) + (2a-2)h - (2b-1)k = 0, \quad g(a, b) + (2b-1)h + (2a-2)k = 0,$$

or by the pair of equations

$$\lambda h - \mu k = -f(a, b), \quad \mu h + \lambda k = -g(a, b),$$

where λ, μ are any two numbers not differing greatly from $2a-2$, $2b-1$. For the root c, we have approximately $a = 0{\cdot}5$, $b = 2{\cdot}5$, and we take $\lambda = -1$, $\mu = 4$; with these values,

$$17h = f - 4g, \quad 17k = 4f + g,$$

accurately, but we do not lose in such rough approximations as these if we replace 17 by 16 and use the still simpler pair of formulae

$$h = \tfrac{1}{4}(\tfrac{1}{4}f - g), \quad k = \tfrac{1}{4}(f + \tfrac{1}{4}g).$$

For the root d, we take $a = 1{\cdot}5$, $b = -1{\cdot}5$, $\lambda = 1$, $\mu = -4$, and by coincidence the pair of formulae for approaching this root is

$$h = -\tfrac{1}{4}(\tfrac{1}{4}f - g), \quad k = -\tfrac{1}{4}(f + \tfrac{1}{4}g).$$

Applying these pairs of formulae to the two roots, taking for first approximations the values in (i), I find for c,

$a_1 =$	$0{\cdot}475$	$f_1 =$	$0{\cdot}0287$	$h_1 =$	$-0{\cdot}0^273$
$b_1 =$	$2{\cdot}370$	$g_1 =$	$0{\cdot}0365$	$k_1 =$	$0{\cdot}0^294$
$a_2 =$	$0{\cdot}4672$	$f_2 =$	$0{\cdot}0^2173$	$h_2 =$	$0{\cdot}0^378$
$b_2 =$	$2{\cdot}3794$	$g_2 =$	$-0{\cdot}0^2269$	$k_2 =$	$0{\cdot}0^326$

$$\begin{array}{lll} h_3 = 0{\cdot}0^4036 & h_4 = -0{\cdot}0^670 & h_5 = 0{\cdot}0^703 \\ k_3 = -0{\cdot}0^4212 & k_4 = 0{\cdot}0^602 & k_5 = 0{\cdot}0^726 \end{array}$$

and I find for d,

$$\begin{array}{lll} a_1 = 1{\cdot}542 & f_1 = 0{\cdot}0169 & h_1 = -0{\cdot}0102 \\ b_1 = -1{\cdot}378 & g_1 = -0{\cdot}0367 & k_1 = -0{\cdot}0019 \\ a_2 = 1{\cdot}5318 & f_2 = -0{\cdot}0^2121 & h_2 = 0{\cdot}0^321 \\ b_2 = -1{\cdot}3799 & g_2 = 0{\cdot}0^2054 & k_2 = 0{\cdot}0^327 \\ h_3 = 0{\cdot}0^574 & h_4 = -0{\cdot}0^630 & h_5 = -0{\cdot}0^705 \\ k_3 = -0{\cdot}0^587 & k_4 = -0{\cdot}0^616 & k_5 = 0{\cdot}0^714. \end{array}$$

Thus to eight places of decimals,

$$c = 0{\cdot}46798290 + 2{\cdot}37963885i,$$

$$d = 1{\cdot}53201710 - 1{\cdot}37963885i.$$

4. The processes described here remain effective even if different roots have the same modulus. The effect of equal moduli on the Gräffe equations is well known. If the modulus r is common to several roots, the two equations in t have several common roots, and in reducing the degree of these equations we reach two equations which are indistinguishable, within the limits of approximation adopted, before the degree reaches unity; each real root of these equations provides an angle to be associated with the modulus r.

In practice it is worth while to calculate f and g at each stage to the full number of decimals that is required in the end, not simply to the number necessary to yield the next approximation; it is then only the differences $f(a+h, b+k) - f(a, b)$, $g(a+h, b+k) - g(a, b)$ that are evaluated, and the leading figures disappear successively from the working.

As soon as h, k are so small that terms of the second order in $f(a+h, b+k)$ and $g(a+h, b+k)$ are literally negligible, we can complete the evaluation in one stage by returning from the approximate pair of equations to the accurate pair, but it must be remembered that to substitute a few heavy calculations for a multitude of trivial ones is not necessarily to lighten the total amount of work done, and that the advantage of the self-correcting property of iterative processes will be lost.

E. H. NEVILLE.

GLEANINGS FAR AND NEAR.

1060. HOW ILLOGICAL!

There still exist good-natured Germans who have not yet woken from their sleep and still say: "Yes, but there are some decent Jews too". If every Nazi knew even one decent Jew, then, as there are 12 million Nazis, there would be 12 million decent Jews in Germany. But here we see how illogical these wiseacres are, for there are at most 6 hundred thousand Jews in Germany.—Curt Rosten, *Das A B C des Nationalsozialismus*, p. 198; translated in *Hail, Hitler!* p. 35. [Per Prof. E. H. Neville.]

A FOCUS-SHARING SET OF THREE CONICS.

By E. H. Neville.

If P is a point of intersection of two conics which have one common focus, then $e_1 \mid PM_1 \mid = e_2 \mid PM_2 \mid$. It follows that if the two directrices cut in O, then OP passes through a second point of intersection, and that if directions of measurement perpendicular to the directrices are assigned, the equations $e_1 \,.\, M_1P = \pm e_2 \,.\, M_2P$ give two of the chords of intersection; we may say that these are the chords associated with the common focus.

Given a triangle ABC, let α, β, γ be three conics, with foci B, C and eccentricity e, with foci C, A and eccentricity f, and with foci A, B and eccentricity g. The directrices of α are two lines u_2, u_3 perpendicular to BC, and if distances from these lines and from the perpendicular bisector of BC, measured in the direction from B to C, are u_2, u_3, u, then $u_2 = u + \frac{1}{2}a/e^2$, $u_3 = u - \frac{1}{2}a/e^2$. The three pairs of common chords obtained by taking the conics two together are

$$fv_1 = \pm gw_1, \quad gw_2 = \pm eu_2, \quad eu_3 = \pm fv_3,$$

that is,

$$fv + gw = \tfrac{1}{2}(b/f - c/g), \; gw + eu = \tfrac{1}{2}(c/g - a/e), \; eu + fv = \tfrac{1}{2}(a/e - b/f),$$

$$fv - gw = \tfrac{1}{2}(b/f + c/g), \; gw - eu = \tfrac{1}{2}(c/g + a/e), \; eu - fv = \tfrac{1}{2}(a/e + b/f).$$

Now the distances u, v, w are connected identically by the relation

$$au + bv + cw = 0.$$

If we determine λ, μ, ν by the condition

$$\lambda(fv + gw) + \mu(gw + eu) + \nu(eu + fv) \equiv au + bv + cw,$$

we have $\quad \lambda = k - a/e, \quad \mu = k - b/f, \quad \nu = k - c/g,$

where $2k = a/e + b/f + c/g$, and with these values,

$$\lambda(b/f - c/g) + \mu(c/g - a/e) + \nu(a/e - b/f) = 0,$$

identically. Similarly if we determine λ_1, μ_1, ν_1 by the condition

$$\lambda_1(fv + gw) + \mu_1(gw - eu) - \nu_1(eu - fv) \equiv au + bv + cw,$$

we have $\quad \lambda_1 = k_1 + a/e, \quad \mu_1 = k_1 - b/f, \quad \nu_1 = k_1 - c/g,$

where $2k_1 = -a/e + b/f + c/g$, and with these values,

$$\lambda_1(b/f - c/g) + \mu_1(c/g + a/e) + \nu_1(-a/e - b/f) = 0.$$

Hence the line $fv_1 + gw_1 = 0$ passes through the intersection of $gw_2 + eu_2 = 0$ and $eu_3 + fv_3 = 0$, and also through the intersection of $gw_2 - eu_2 = 0$ and $eu_3 - fv_3 = 0$. That is to say:

If three conics are such that each pair selected from them has one common focus, the pairs of common chords associated with the common pair are the three pairs of sides of a quadrangle.

If β and γ are ellipses, v_1 is negative at every real point of β and w_1 is positive at every real point of γ; hence there can not be any real points of intersection of the ellipses on the line $fv_1 - gw_1 = 0$, and

if the ellipses do intersect, the chord of visible intersection is $fv_1 + gw_1 = 0$.

Two ellipses with a common focus can not have more than two real points of intersection ; if three ellipses are such that each pair has one common focus and two real points of intersection, the three chords of visible intersection are concurrent.

This theorem, from which the present note had its origin, was conjectured by J. S. Turner in 1913 on the basis of careful drawings. Search for a proof at the time failed ; sixteen years later, verification by algebra too tiresome for reproduction, due in part to J. L. Coolidge, enabled the discoverer to communicate the result as an established theorem to the American Association for the Advancement of Science at a meeting in Iowa. As far as Prof. Turner knows, no proof has hitherto been published.

If γ is a hyperbola, w_1 is negative at every real point of the branch nearer to A and positive at every real point of the other branch ; hence if β is an ellipse and γ is a hyperbola, β can not cut the nearer branch of γ except on the line $fv_1 - gw_1$ and can not cut the farther branch of γ except on the line $fv_1 + gw_1 = 0$. Since β can not cut the farther branch without cutting the nearer branch also, it follows that if there are two and only two real points of intersection, the line through them is $fv_1 - gw_1 = 0$.

If β as well as γ is a hyperbola, v_1 is positive on the branch nearer to A and negative on the other branch. Hence a real point of intersection on $fv_1 - gw_1 = 0$ is a point in which the nearer branch of one hyperbola cuts the farther branch of the other, and a real point on $fv_1 + gw_1 = 0$ is a point common either to the two nearer branches or to the two farther branches. But it is easily proved that if the nearer branch of β cuts the farther branch of γ, either the nearer branch of β cuts the nearer branch of γ also or the farther branch of β also cuts the farther branch of γ ; that is to say, if there are real points of intersection on $fv_1 - gw_1 = 0$ there must be real points on $fv_1 + gw_1 = 0$ also, and from this it follows that if there are two and only two real points of intersection, these are on $fv_1 + gw_1 = 0$.

Thus we can complete Turner's theorem :

If three conics are such that each pair has one common focus and two and only two real points of intersection, the three chords of visible intersection are concurrent.

But since the six sides of the quadrangle in our first theorem are always real, while the number of chords of visible intersection may have any value from zero to six, the earlier theorem is of more interest even practically. Theoretically there is no comparison, for the one can be extended by projection, reciprocation, and other transformations, while the other can not. One version of the more general theorem is the following :

If α, β, γ are the sections of a conicoid by three planes whose common point is in the surface, the vertices of the two cones through β and γ, of the two cones through γ and α, and of the two cones through α and β, are the six vertices of a quadrilateral.

E. H. N.

DUAL VECTORS AND THE PETERSEN-MORLEY THEOREM.

BY J. A. TODD.

SEVERAL proofs, based on projective considerations, have been given recently of the Petersen-Morley theorem.† In this note a direct metrical proof is given, based on the dual vectors of Study, which seems simpler than any of these. The theorem in question may be stated as follows :

If a, b', c, a', b, c' are six lines in space forming a skew hexagon, and if all the angles between adjacent sides of the hexagon are right angles, then the three lines p, q, r drawn respectively to meet a, a' ; b, b' ; c, c' at right angles have a common perpendicular transversal.

If $\mathbf{a}_1$, $\mathbf{a}_2$ are real three-component vectors, and ϵ is a scalar multiplier such that $\epsilon^2=0$, the expression $\mathbf{A}=\mathbf{a}_1+\epsilon\mathbf{a}_2$ is called a *dual vector*, and we may refer to $\mathbf{a}_1$ as its *real part*. For these dual vectors we can define vector and scalar products in an obvious way ; thus the vector product of $\mathbf{A}$ and the dual vector $\mathbf{B}=\mathbf{b}_1+\epsilon\mathbf{b}_2$ is the dual vector $\mathbf{A}\times\mathbf{B}=\mathbf{a}_1\times\mathbf{b}_1+\epsilon(\mathbf{a}_1\times\mathbf{b}_2+\mathbf{a}_2\times\mathbf{b}_1)$, and the scalar product of $\mathbf{A}$ and $\mathbf{B}$ is the dual number $\mathbf{A}.\mathbf{B}=\mathbf{a}_1.\mathbf{b}_1+\epsilon(\mathbf{a}_1.\mathbf{b}_2+\mathbf{a}_2.\mathbf{b}_1)$. Dual vectors obey the ordinary rules of vector algebra ; in particular we have

$$\mathbf{A}.\mathbf{B}=\mathbf{B}.\mathbf{A}\,;\quad \mathbf{A}\times\mathbf{B}=-\mathbf{B}\times\mathbf{A}\,;\quad \mathbf{A}\times\mathbf{A}=0.$$

If the vector $\mathbf{a}_1$ is not null we can find a unique real scalar h such that $\mathbf{a}_1.(\mathbf{a}_2+h\mathbf{a}_1)=0$. Thus any dual vector whose real part is different from zero determines another dual vector $\mathbf{A}^*=\mathbf{a}_1+\epsilon\mathbf{a}_2^*$ with the same real part, such that $\mathbf{a}_1.\mathbf{a}_2^*=0$. Such a dual vector may be called *orthogonal*. The six components of the constituent vectors of an orthogonal dual vector may be taken to be the Plücker coordinates of a line in Euclidean space, referred to a tetrahedron consisting of three mutually perpendicular planes and the plane at infinity, and conversely any such line defines an orthogonal dual vector, in which the real part has components proportional to the direction-cosines of the line.

In the applications of the calculus of dual vectors to line-geometry made by Study ‡ only orthogonal dual vectors come into play. For our present purpose it is convenient to remove this restriction. A line in Euclidean space may be represented equally well by any one of a family of dual vectors obtained from some orthogonal dual vector $\mathbf{a}_1+\epsilon\mathbf{a}_2$ by adding any real scalar multiple of $\epsilon\mathbf{a}_1$, or by multiplying by a real scalar (*i.e.* we deal with homogeneous vectors). Each such dual vector with a non-vanishing real part determines an associated orthogonal dual vector to within a real scalar factor, and

† See, *e.g.* H. F. Baker, *Jour. London Math. Soc.*, 11 (1936), 24 ; or R. Frith, *Proc. Camb. Phil. Soc.*, 30 (1934), 192, 197.

‡ Cf. Blaschke, *Differentialgeometrie*, i (2 ed. Berlin, 1924), 191.

hence determines a line in space. The utility of this representation rests on the two following propositions :

I. If **A**, **B** are dual vectors corresponding to two lines a, b then the vanishing of the scalar product $\mathbf{A} \cdot \mathbf{B}$ is the necessary and sufficient condition that a and b should intersect at right angles.

II. If **A**, **B** are dual vectors representing two non-parallel lines a, b then the dual vector $\mathbf{A} \times \mathbf{B}$ represents the common perpendicular transversal of a and b.

To prove I, let $\mathbf{A}^*$, $\mathbf{B}^*$ be the orthogonal dual vectors corresponding to **A** and **B**. It is known that the condition that a, b should cut at right angles is that $\mathbf{A}^* \cdot \mathbf{B}^* = 0$.† That this is equivalent to $\mathbf{A} \cdot \mathbf{B} = 0$ follows immediately from the fact that $\mathbf{A}^* - \mathbf{A} = h\epsilon\mathbf{a}_1$, $\mathbf{B}^* - \mathbf{B} = k\epsilon\mathbf{b}_1$, where h, k are real scalars and $\mathbf{a}_1$, $\mathbf{b}_1$, are the real parts of **A**, **B**, and from the fact that $\mathbf{a}_1 \cdot \mathbf{b}_1 = 0$. The result II is an immediate consequence of I, since the scalar product of $\mathbf{A} \times \mathbf{B}$ with **A** or **B** vanishes. A corollary is that the necessary and sufficient condition that three lines a, b, c possess a common transversal perpendicular is that the triple scalar product $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ of the corresponding dual vectors should vanish.

We can now give a very simple proof of the Petersen-Morley theorem. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{A}', \mathbf{B}', \mathbf{C}', \mathbf{P}, \mathbf{Q}, \mathbf{R}$ be dual vectors corresponding respectively to $a, b, c, a', b', c', p, q, r$. Since a' meets b and c at right angles we may take $\mathbf{A}' = \mathbf{B} \times \mathbf{C}$; similarly $\mathbf{B}' = \mathbf{C} \times \mathbf{A}$ and $\mathbf{C}' = \mathbf{A} \times \mathbf{B}$. Since p meets a and a' at right angles we may take $\mathbf{P} = \mathbf{A} \times \mathbf{A}' = \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$. Similarly $\mathbf{Q} = (\mathbf{B} \cdot \mathbf{A})\mathbf{C} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$; $\mathbf{R} = (\mathbf{C} \cdot \mathbf{B})\mathbf{A} - (\mathbf{C} \cdot \mathbf{A})\mathbf{B}$. Hence $\mathbf{P} + \mathbf{Q} + \mathbf{R} = 0$ and so $\mathbf{P} \cdot \mathbf{Q} \times \mathbf{R} = -\mathbf{P} \cdot \mathbf{Q} \times (\mathbf{P} + \mathbf{Q}) = 0$. Hence the lines p, q, r have a common perpendicular transversal, which proves the theorem.

We can obtain other results in the same way. Thus we can prove that, if a, b, c, d are four skew lines such that the common perpendicular transversals of a, c, and of b, d meet at right angles, and also the common perpendicular transversals of a, d, and of b, c meet at right angles, then the common perpendicular transversals of a, b, and of c, d also meet at right angles. The reader will easily construct the proof of this theorem for himself, starting from the identity $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$. J. A. T.

1061. The law of gravitation, according to Eddington, is only a convenient convention of measurement. It is not truer than other views, any more than the metric system is truer than feet and yards.

'Nature and Nature's laws lay hid in night.
God said, "Let Newton be", and measurement was facilitated'.

This sentiment seems lacking in sublimity. When Spinoza believed anything, he considered that he was enjoying the intellectual love of God. The modern man believes either with Marx that he is swayed by economic motives, or with Freud that some sexual motive underlies his belief in the exponential theorem or in the distribution of fauna in the Red Sea. In neither case can he enjoy Spinoza's exaltation.—B. Russell, *In Praise of Idleness*, 1935, p. 183. [Per Prof. E. H. Neville.]

† Blaschke, *loc. cit.* 194. The result may easily be verified directly.

ON A THEOREM IN HIGHER PLANE CURVES.

By Clifford Bell.

In Hilton's *Plane Algebraic Curves* the functions $F(t)$ and $F'(t)$ are used in finding inflexions and cusps of plane curves. These functions are defined, respectively, by the determinants

$$\begin{vmatrix} f(t) & \phi(t) & \psi(t) \\ f'(t) & \phi'(t) & \psi'(t) \\ f''(t) & \phi''(t) & \psi''(t) \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} f(t) & \phi(t) & \psi(t) \\ f'(t) & \phi'(t) & \psi'(t) \\ f'''(t) & \phi'''(t) & \psi'''(t) \end{vmatrix}$$

where $f(t)$, $\phi(t)$, $\psi(t)$ are the values, in terms of a parameter t, of the homogeneous coordinates of a point on the curve. It is shown that $F(t_1)$ vanishes if t_1 is the parameter of an inflexion or a cusp and $F'(t_1)$, the derivative of $F(t_1)$, vanishes if t_1 gives a cusp.

Assume that in general to each point on the curve corresponds a single value of t and that the curve under consideration has only ordinary singularities. Under these conditions Hilton states, if f, ϕ, ψ are polynomials in t, the cusps are given by the repeated roots of $F(t)=0$ and the inflexions by the single roots.

Although this proposition may be proven in other ways, the author of this note offers the following proof as an illustration of the usefulness of the Plücker numbers.

As the parameter of each inflexion and of each cusp satisfies $F(t)=0$, it is evident that amongst the roots of $F(t)=0$ are found the parameters of all the inflexions and cusps. Also, as the parameter of each cusp satisfies $F'(t)=0$, it follows that the parameter of each cusp is at least a double root of $F(t)=0$. It remains to be shown that $F(t)=0$ has no roots other than the parameters of inflexions and cusps. Also, if possible, the exact multiplicity of the roots that give cusps will be found.

Let the polynomials f, ϕ, ψ be represented, respectively, by

$$a_0t^n + a_1t^{n-1} + \ldots + a_n, \quad b_0t^n + b_1t^{n-1} + \ldots + b_n, \quad c_0t^n + c_1t^{n-1} + \ldots + c_n,$$

where not all of the constants a_0, b_0, c_0 are zero. Using these values the first column of the $F(t)$ determinant becomes

$$\begin{array}{l} a_0t^n + a_1t^{n-1} + \ldots + a_n \\ na_0t^{n-1} + (n-1)\, a_1t^{n-2} + \ldots + a_{n-1} \\ n(n-1)\, a_0t^{n-2} + (n-1)(n-2)\, a_1t^{n-3} + \ldots + a_{n-2}, \end{array}$$

while the second and third columns are obtained by the substitution of b and c, in turn, for a. Making use of the elementary theorems on determinants, this determinant may be transformed into the determinant whose first column is

$$\begin{array}{l} 2a_2t^{n-2} + \ldots \\ a_1t^{n-2} + \ldots \\ a_0t^{n-2} + \ldots, \end{array}$$

and again the second and third columns are obtained by the sub-

stitution of b and c for a. The maximum degree of this determinant is readily seen to be $3n-6$.

As the coordinates of the curve in question are represented by means of polynomials, the deficiency of the curve is zero, and hence

$$\delta+\kappa=\tfrac{1}{2}(n-1)(n-2) \tag{1}$$

where n is the order of the curve, δ the number of nodes, and κ the number of cusps. Letting ι represent the number of inflexions, the Plücker equation giving ι is

$$\iota=3n(n-2)-6\delta-8\kappa. \tag{2}$$

Eliminating δ between equations (1) and (2) it is found that $\iota+2\kappa=3n-6$, which means that $\iota+2\kappa$ is equal to the maximum degree in t of $F(t)=0$.

Now as the parameter of a cusp must appear at least as a double root of $F(t)=0$, the minimum degree in t of $F(t)$ is $\iota+2\kappa$, where infinite values are counted as possible parameters of cusps or inflexions. Hence, as $\iota+2\kappa$ has been shown to be both the minimum and maximum degree of $F(t)=0$, it follows that the degree in t of $F(t)=0$ is exactly $\iota+2\kappa$ or $3n-6$, infinite roots being counted.

Thus the objectives previously stated have been accomplished, namely: $F(t)=0$ has no roots other than those that give inflexions and cusps, and the parameters of the cusps appear as double roots of $F(t)=0$. C. B.

GOVERNMENT REPRESENTATIVES TO N.E.F. CONFERENCE.

THE Board of Education in England and Wales has appointed as its two delegates to the Seventh World Conference of the New Education Fellowship, which will be held at Cheltenham in the first fortnight of August to discuss "Education and a Free Society", His Majesty's Inspectors, Mr. R. D. Salter Davies and Mr. P. G. Coles. The Scottish Education Department has appointed Mr. G. Andrews, His Majesty's Chief Inspector of Schools, to represent it. Denmark, Northern Ireland and France have also appointed official Government representatives.

Besides 300 United States delegates who are travelling to the Conference on board the *Berengaria*, a party of Americans touring Europe under the auspices of the Teachers' College, Columbia University, will make the Conference one of its features. The Norwegian Teachers' Association has notified its intention of sending a party, and from half-way across the world will be a delegation from Japan and another from India.

So wide is the appeal of the subject that the Conference is being thrown open to all, and Bedford Training College Old Students' Association is to hold its re-union at Cheltenham at the time of the Conference so that its members may attend.

Mr. R. R. Tomlinson, Director of the Central School of Arts in London, has been appointed organiser of the International Arts Exhibition of the Conference. To the distinguished panel of internationalists who will lecture at Cheltenham have been added the names of two celebrated oriental scholars, Sir Savepalli Radhakrishnan, India, and Dr. Peng-Chun Chang, China.

VECTOR NOTATION

WITH A TRANSLATION OF THE "NORMBLÄTT" OF THE AUSSCHUSS FÜR EINHEITEN UND FORMELGRÖSSEN OF THE DEUTSCHER NORMENAUSSCHUSS.

BY G. WINDRED.

FROM the first inception of vector analysis, the subject of notation has engaged the attention of many writers, but up to the present there has, unfortunately, been no sign of general agreement upon a standard notation.

It would be difficult to overestimate the importance of this matter. The history of mathematics abounds with examples of advancements which may be attributed, directly or indirectly, to improvements in notation, and in the interests of progress it is to be hoped that some agreement, not only national but international, will before long be forthcoming

Considerable difficulty arises from the necessity of distinguishing between the scalar (internal) and the vector (external) product of vectors and it is interesting to compare the notations of different writers, as follows :

	Scalar Product.	*Vector Product.*
Gans	$(\mathbf{a} \,.\, \mathbf{b})$	$[\mathbf{a} \,.\, \mathbf{b}]$
Gibbs	$\mathbf{a} \,.\, \mathbf{b}$	$\mathbf{a} \times \mathbf{b}$
Hamilton	$S\mathbf{ab}$ or $S\alpha\beta$	$V\mathbf{ab}$ or $V\alpha\beta$
Heaviside	$\mathbf{ab}$	$V\mathbf{ab}$
Runge	$\mathbf{a} \,.\, \mathbf{b}$	$\mathbf{a} \times \mathbf{b}$ or $\mathbf{ab}$

This represents not only great diversity of opinion as to what constitutes a good notation, but also the great risk of confusion which exists.

The problem has received detailed attention in Germany, where the Ausschuss für Einheiten und Formelgrössen (AEF) of the Deutscher Normenausschuss has issued an important standard publication (Normblätt) on the subject. A translation of this is reproduced herewith by kind permission of the Deutscher Normenausschuss : it may well serve as the basis of negotiations in this country with a view to establishing a much-needed standard vector notation.

TRANSLATION OF DIN 1303 "VEKTORZEICHEN."*

(Vector Symbols)

1. Vectors are represented by (small or large) letters : $\mathbf{a}$, $\mathbf{b}$, ... $\mathbf{A}$, $\mathbf{B}$, ...

Supplement 1. In special cases vectors can be indicated by a stroke thus : $\bar{r}$, $\bar{\omega}$.

* It will be understood that in the original all vectors are represented by German (Fraktur) letters. A new edition of DIN 1303 is in course of preparation.

Supplement 2. If a vector be specified by its starting point A and ending point B, it may be represented by $\overrightarrow{AB}$.

2. The magnitude of the vector **A** is represented by $|\mathbf{A}|$; and, where no misunderstanding is likely, also by A.

3. The unit vector corresponding to the vector A is represented by $\mathbf{A}^\circ$, so that $\mathbf{A} = |\mathbf{A}|\,\mathbf{A}^\circ$. In special cases the unit vector can also be represented by the corresponding small letter: $\mathbf{A} = |\mathbf{A}|\,\mathbf{a}$.

4. The usual plus and minus signs serve to indicate vector sums and differences: $\mathbf{A}+\mathbf{B}$, $\mathbf{A}-\mathbf{B}$.

5. The scalar product of two vectors is represented by writing the respective symbols together: **AB**. Where necessary, vector sums, vector differences and scalar products are to be enclosed in round brackets: $\mathbf{A}(\mathbf{B}+\mathbf{C})$, $(\mathbf{AB})\mathbf{C}$. Scalar factors can also be separated by a point: $\mathbf{AB}\,.\,\mathbf{C} = (\mathbf{AB})\mathbf{C}$.

Supplement. $\mathbf{A}^2$ represents the scalar product of **A** with itself.

6. The vector product of two vectors is represented by writing the respective symbols together and enclosing them in square brackets: $[\mathbf{AB}]$, $[\mathbf{A}(\mathbf{B}+\mathbf{C})]$.

Supplement. The product $\mathbf{A}[\mathbf{BC}]$ can be expressed by **ABC**.

7. For the gradient of the scalar ϕ, the divergence and rotation of the vector **A**, the symbols grad ϕ, div **A**, rot **A** are used.

Supplement 1. The rotation of the rotation of **A** is represented by rot rot **A**.

Supplement 2. If the gradient of the scalar product **AB** be required, wherein only **A** is variable and **B** is constant, the operation is expressed by $\text{grad}_{\mathbf{A}}(\mathbf{AB})$.

8. The Hamiltonian operator is represented by ∇ (pronounced Nabla). When applied to gradients, divergences and rotations, the symbols given in 7 above are preferred, and ∇^2 can be replaced by Δ.

Supplement. If in $\nabla(\mathbf{AB})$ only **A** is variable and **B** is constant, the operation is expressed by $\nabla_{\mathbf{A}}(\mathbf{AB})$. The same applies in corresponding cases with other products involving ∇.

9. The differences involved in the conceptions of grad ϕ, div **A** and rot **A** in the case of surfaces of discontinuity are denoted by Grad ϕ, Div **A**, Rot **A** (large initials).

10. The fundamental vectors (three unit vectors at right angles to each other) are denoted by **i**, **j**, **k**.

11. The vectorial projection of a vector **A** on another vector **B** is indicated by $\mathbf{A}_{\mathbf{B}}$, and the vectorial projection of a vector **A** on a line x is indicated by $\mathbf{A}_x$. The scalar components referred to rectangular axes are A_x, A_y, A_z.

When the directions of **i**, **j**, **k** correspond with those of the axes x, y, z, then

$$\mathbf{A}_x = A_x\mathbf{i} \qquad \mathbf{A}_y = A_y\mathbf{j} \qquad \mathbf{A}_z = A_z\mathbf{k}$$

$$\mathbf{A} = \mathbf{A}_x + \mathbf{A}_y + \mathbf{A}_z = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}.$$

Supplement. In certain cases the scalar components are indicated by other suitable Latin or Greek letters :

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad \overline{\omega} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k}.$$

G. Windred.

MISS L. M. SWAIN

It is with very great regret that we record the death of Miss Lorna M. Swain, Fellow and Lecturer of Newnham College, Cambridge, and a member of the Mathematical Association since 1920.

Her life as a University lecturer, member of a Faculty Board and Director of Mathematical Studies at Newnham was a busy one, yet she readily added to her work an active share in that of the Mathematical Association. She was elected to the Girls' Schools Committee in 1923 and was one of the original members of the General Teaching Committee constituted in 1923 ; she was a member of the sub-committee on Mechanics which, in 1930, produced the Association's report on the teaching of mechanics ; she also shared in the production of the report on mathematics in girls' schools. She became a member of the Council in 1927, retiring by rotation in 1931, and being re-elected in 1935. She was still serving on these three bodies at the time of her death, and also as secretary of the committee which draws up the programme for the annual meeting of the Association.

Whatever she undertook was carried out quietly and effectively. In discussion, though never anxious to put her own views forward, she was always ready to give to the committee the results of her knowledge and experience and her clear thinking and unbiassed judgment, combined with her impersonal outlook, were of great value. Mathematical teaching in girls' schools owes much not only to the scholarly and thorough training she gave her students but also to her influence on their after careers, for she took great trouble to keep in touch with them and readily gave them help and advice when asked.

Modest about her own achievements and appreciative of those of others, generous in thought and action, naturally reserved and dignified, yet friendly and understanding, she will be much missed by all who knew her, old and young. To her mother we would like to offer our deep sympathy and to express the gratitude of the Association for the service her daughter so ably gave it.

M. J. G.

THEORY OF THE GENERAL FRÉGIER-POINT.*

By A. A. Krishnaswami Ayyangar.

§ 1. It is proposed to discuss in this paper some properties of the general Frégier-point and point out its connections with well-known conics like the Apollonian hyperbola and the nine-point conic of a quadrangle. Incidentally, we discover a *pedal conic* of a quadrangle which has very intimate relations with the nine-point conic.

Given a point P and a conic " S " with centre O, the harmonic conic-envelope of " S " and the circular lines through P may be denoted by ${}_S\Phi_P$. Tangents to ${}_S\Phi_P$ are chords of " S " which subtend a right-angle at P, which is also a focus of ${}_S\Phi_P$. The conjugate focus Q of ${}_S\Phi_P$ is sometimes called the general Frégier-point, which becomes the usual Frégier-point when P lies on " S ". We shall, however, call (P, Q) an isoclinal couple, or more briefly an *isocouple* of " S " ; P is the *left isoclinal point* of Q, and Q *the right isoclinal point* of P with respect to " S ". The reasons for this nomenclature will be found in the sequel. (*Vide* Theorem VII *infra.*)

If P lies on " S ", its right isoclinal point is the Frégier-point, while its left isoclinal point is the common point of the pedal circles of P with respect to triangles inscribed in " S ". (*Vide* Th. I, Cor. 3, and H. F. Baker's *Principles of Geometry*, vol. ii, p. 87.)

§ 2. *Conics having a given isocouple* (P, Q).

Theorem I. P is a given point, I, J are the circular points at infinity whose polars with respect to a given conic " S " cut PJ, PI in L, M respectively, and LI, MJ meet in Q. Then Q is the right isoclinal point of P with respect to " S."

Proof. (I, L) being a conjugate pair with respect to " S ", LI touches ${}_S\Phi_P$, and so also MJ.

Similarly, if the polars of I, J with respect to " S " cut PI, PJ respectively in L', M', $L'I$, $M'J$ also touch ${}_S\Phi_P$. Hence (P, Q, L, M) are the four foci of ${}_S\Phi_P$; Q, being the focus conjugate to P, is the right isoclinal point of P with respect to " S ".

N.B.—This theorem enables us to construct the right and the left isoclinal points of a given point.

Cor. 1. If LM, IJ meet in T and TP cut LI, MJ in U, V respectively, PQ is the perpendicular bisector of LM, UV.

Cor. 2. P, T are conjugate points with respect to " S ". The polar of P is perpendicular to PQ.

Cor. 3. If Q lies on " S ", since $(IL, QU) = -1$, and (I, L) are conjugate with respect to " S ", U must lie on " S " and similarly V. Hence, if the circular lines through Q, any point on " S ", meet " S " again in U, V, the middle point of UV, which is also the foot of the perpendicular from Q on UV, is the left isoclinal point of Q with respect to " S ". It is well known that the foot of the perpen-

* I am indebted to Mr. V. Ramaswami Aiyar for the substance of the theorems marked with an asterisk (✱). For the proofs, however, I am entirely responsible.

dicular from Q on UV is the common point of the pedal circles of Q with respect to the inscribed triangles of " S ". (*Vide* H. F. Baker, *l.c.*)

COR. 4. Any isocouple of " S " is also an isocouple for $S+\lambda\mathcal{L}^2=0$ where $\mathcal{L}=0$ is the equation of the line at infinity and λ any constant.

Thus, *any isocouple* of " S " *is also an isocouple of its asymptotes and vice versa.* *

COR. 5. The left isoclinal point of any point with respect to a rectangular hyperbola is the centre of the hyperbola, and the right isoclinal point of any point with respect to a circle is the centre of the circle.

Theorem II. If S_1, S_2, S_3 be any three linearly independent conics for which a given pair of points (P, Q) is an isocouple, then all the conics having the same isocouple are included in the linear three-parameter system

$$\lambda_1 S_1+\lambda_2 S_2+\lambda_3 S_3+\lambda_4\mathcal{L}^2=0,$$

where $\mathcal{L}=0$ is the line at infinity.

This theorem follows immediately from two facts, viz.: (1) Conics with a given isocouple have two given pairs of points as conjugate pairs and therefore have three degrees of freedom; (2) two points conjugate with respect to two or more conics are also conjugate with respect to a linear system of conics determined by them.

Note. We may conveniently take for S_1, S_2 two orthogonal line-pairs through P, and for S_3 a pair of circular lines through Q. For all rectangular hyperbolas of the system $\lambda_3=0$, and for all circles of the system $\lambda_1=\lambda_2=0$. If T_1, T_2 be any two conics of the system, the conic $T_1+\mu T_2$ also belongs to the system and becomes a rectangular hyperbola (centre P) for a unique value of μ. The locus of the centres of conics $T_1+\mu T_2$ contains the points P, L, M (in the notation of Theorem I), and the middle point of PQ is the left isoclinal of P with respect to this centre-locus.*

Theorem III. Two given conics have a unique common isocouple.

Proof. If P be the centre of the rectangular hyperbola through the four common points of the given conics, and Q the right isoclinal point of P with respect to one of the given conics, then (P, Q) is an isocouple for the rectangular hyperbola also, and therefore for all conics through the four common points. Hence (P, Q) is an isocouple for the two given conics.

If (P', Q') be any other common isocouple for the given conics, then it must also be an isocouple for the rectangular hyperbola through the common points of the two conics. Hence, by Theorem I, Cor. 5, P' coincides with P, and therefore Q' also must coincide with Q.

COR. Every four-point system of conics has a unique common isocouple.

Theorem IV. The locus of the right isoclinal points of a given point P with respect to a four-point system of conics is, in general,

a straight line ; * and the corresponding locus for the left isoclinal points is in general a circle, but degenerates into a straight line when the four-point system includes a circle.

Proof. The polars of I, J, the circular points at infinity with respect to the given system of conics, cut PJ, PI respectively in two homographic ranges of points, and the joins of pairs of corresponding points of these ranges envelop a conic, say Γ. In the notation of Theorem I, if L, M be the points of incidence of PJ, PI with the polars of I, J with respect to a conic of the four-point system, it is easily seen that LM, PI, PJ, IJ are joins of pairs of corresponding points of the homographic ranges aforesaid. Hence these lines touch Γ, which is a parabola with P as the focus. The middle point of LM, being the foot of the perpendicular from P on LM, lies on the tangent at the vertex of the parabola Γ. The locus of the right isoclinal point of P, which is the symmetrique of P in the middle point of LM is the directrix of the parabola Γ.

The discussion for the left isoclinal points of P proceeds on exactly similar lines, with the difference that I, J are not corresponding points of the homographic ranges unless the four-point system of conics contains a circle. Thus the conic-envelope Γ' (say) corresponding to Γ is not a parabola, and the middle point of LM lies on the auxiliary circle of Γ'. Hence the locus of the left isoclinal points of P is also a circle whose centre is the other focus of Γ'.

In the exceptional case, however, when the given four-point system of conics includes a circle, Γ' becomes a parabola, and the locus of the left isoclinal points becomes the directrix of Γ'.

Exceptions. When P is the centre of the rectangular hyperbola of the four-point system of conics, its right isoclinal point is the same for all conics of the system.

If the left isoclinal point of P for some conic of the system is the centre of the rectangular hyperbola of the system, this left isoclinal point is the same for all conics of the system.

The loci in these cases dwindle to a point.

Theorem V. The left isoclinal loci of points on a given straight line, with respect to a four-point system of conics, form a co-axal system of circles.

Proof. Let M be the centre of the rectangular hyperbola belonging to the four-point system of conics; and let (M, N) be the common isocouple of the four-point system.

If the given straight line does not pass through N, take three points P, Q, R on it and denote their left isoclinal loci by C_P, C_Q, C_R, which are evidently circles passing through M.

Let the circles C_P, C_Q intersect again in P' ; then the right isoclinal locus of P' with respect to the given system of conics is the straight line PQ, and so (P', R) is an isocouple for some conic of the four-point system. Hence C_R also passes through P' and C_P, C_Q, C_R are co-axal.

If the given straight line passes through N, P' coincides with M, and the circles C_P, C_Q, C_R touch one another at M.

§ 3. *Metrical properties of an isocouple.*

Theorem VI. If (P, Q) be an isocouple with respect to a line-pair $T \equiv (OA, OB)$, then

(i) the perpendicular from P on OA (or OB) passes through the foot of the perpendicular from Q on OB (or OA);

(ii) OP, OQ are equally inclined to (OA, OB); and

(iii) $OP = OQ \cos A\hat{O}B$.

Proof. Let the perpendicular from P on OA meet OB in K. Through K draw KK' parallel to OA. We may now regard KO, KK' as tangents to the conic ${}_T\Phi_P$, of which P, Q are a pair of real foci. Hence, from the angle-property of tangents to a conic,

$$P\hat{K}O = Q\hat{K}K',$$

so that
$$Q\hat{K}B = Q\hat{K}K' + K'\hat{K}B = P\hat{K}O + K\hat{O}A = \tfrac{1}{2}\pi.$$

Hence K is the foot of the perpendicular from Q on OB. Similarly, if the perpendicular from P on OB meet OA in H, QH is perpendicular to OA.

Thus, the four points O, P, H, K form an orthocentric tetrad, Q is the point diametrically opposite to O in the circumcircle of the triangle OHK, and PQ, HK bisect each other.

By elementary geometry, $OP = OQ \cos AOB$ and (OP, OQ) are equally inclined to (OA, OB).

N.B. If one of the points (P, Q) goes off to infinity, the other point also goes to infinity, except when $A\hat{O}B = \frac{1}{2}\pi$, in which case it becomes indeterminate.

By Theorem I, Cor. 4, an isocouple of a line-pair is also an isocouple with respect to any conic having the line-pair as its asymptotes; hence,

Theorem VII.* The lines joining the centre O of a conic to an isocouple (P, Q) are equally inclined to the asymptotes and to each of the axes of the conic; the ratio $OP : OQ$ is equal to the cosine of the angle between the asymptotes.

If X, Y be the feet of the perpendiculars from Q on the asymptotes, then PQ, XY bisect each other.

N.B. The above angle-properties of OP, OQ have suggested the designation "isocouple".

Cor. 1. If (P, Q) be an isocouple with respect to a parabola, the middle point of PQ lies on the axis of the parabola.

Cor. 2. If (P_1, Q_1), (P_2, Q_2) be two isocouples with respect to a conic, the triangles OP_1P_2, OQ_1Q_2 are inversely similar. Hence, on every straight line other than an axis of a conic, there is only one isocouple. On any diameter other than the axes, the only isocouple

* This theorem is not exactly new. The substance of it can be traced to J. Wolstenholme's *Mathematical Problems*, p. 211, Q. 1267.

is the pair of points coinciding with the centre; while on an axis, we have an infinite number of isocouples.

Cor. 3. If (P_1, Q_1), (P_2, Q_2), (P_3, Q_3) be three isocouples with respect to a conic and (P_1, P_2, P_3) are collinear, then (Q_1, Q_2, Q_3) are also collinear, and *vice versa*; and the two lines P_1P_2 and Q_1Q_2 are equally inclined to each of the axes of the conic.

Cor. 4. Given the centre O of a conic and an isocouple (P, Q), the asymptotes of the conic are thereby fixed.

Again, given one asymptote and an isocouple of a conic, the other asymptote also gets fixed.

Theorem VIII. If X, Y be the feet of the perpendiculars from a given point P on the asymptotes of a conic of a four-point system, the locus of the middle point of XY is, in general, a straight line or a circle, according as the four-point system does or does not include a circle. If for some conic of the system, P becomes the right isoclinal point of the centre of the rectangular hyperbola belonging to the four-point system, the middle point of XY becomes a fixed point. [This follows readily from Theorems IV and VII.]

§ 4. *The Apollonian hyperbola through an isocouple.*

The feet of the normals from any point P to a conic "S", centre O, lie on a rectangular hyperbola, known as the Apollonian hyperbola, which passes through P and O and has its asymptotes parallel to the axes of "S". Denoting this hyperbola by ${}_SH_P$, it is easily seen that it can be generated as the locus of a point T such that TP is perpendicular to the polar of T with respect to "S". Hence Q, the left isoclinal point of P, lies on ${}_SH_P$; * and since OP, OQ are equally inclined to either axis of "S", and therefore to either asymptote of ${}_SH_P$, PQ must be a diameter of ${}_SH_P$. Hence,

Theorem IX. The left isoclinal point of a given point with respect to a conic is diametrically opposite to it in the Apollonian hyperbola pertaining to the given point.

Theorem X. If (OA, OB), a pair of conjugate diameters of "S", cut ${}_SH_P$ in (A, B), then P is the orthocentre of the triangle OAB whose circumcircle passes through the left isoclinal point of P.

For the polar of A is parallel to OB, which is therefore perpendicular to PA, and similarly PB is perpendicular to OA. The rest of the theorem is an easy deduction from the properties of the rectangular hyperbola.

Theorem XI. If (P, Q) be an isocouple of "S" and P trace any straight line l, then Q traces another straight line m such that (l, m) are equally inclined to an axis of "S", and the locus of the intersection of (l, m) as l turns about P as a fixed point is ${}_SH_Q$.

The proof of this follows from Theorem VII, Cor. 3, and Theorem IX.

N.B. We call m the right isoclinal line of l, and l the left isoclinal line of m.

COR. 1. The right isoclinal line of PQ is the tangent at Q to ${}_SH_Q$.

COR. 2. If Q is any point on " S ", the normal to " S " at Q is the right isoclinal line of PQ. If PQ and the normal at Q meet " S " again in A, B respectively, the circle ABQ touches " S " at Q and $QAB=\frac{1}{2}\pi$.

Theorem XII. If (P, Q) be an isocouple " S " which passes through Q, and the circle on PQ as diameter cut " S " again in A, B, C, and PA, PB, PC cut " S " in X, Y, Z respectively, then the normals at X, Y, Z concur at Q.

Proof. As remarked at the end of § 1, the pedal circle of Q with respect to any inscribed triangle of " S " passes through P.

If the circle AXQ cut " S " again in A', the pedal-line of Q with respect to the triangle AXA' is easily seen to be AP itself, so that QX must be perpendicular to $A'X$ and QA perpendicular to AA'. Therefore A' must coincide with X. This shows that the circle AXQ touches " S " at X, and so XA, XQ are equally inclined to an axis of " S " and an asymptote of ${}_SH_Q$ and X must lie on ${}_SH_Q$. Similarly Y and Z lie on ${}_SH_Q$.

Theorem XIII. If (P, Q), (Q, R) be two isocouples of a conic " S " which passes through Q, and any straight line through P cuts " S " in A, B, while the bisectors of the angle AQB meet " S " again in X, Y, then

(1) R is the middle point of XY;

(2) XY is perpendicular to PQ;

(3) the tangents at X, Y to " S " meet in P; and

(4) QX, QY are equally inclined to an axis of " S ".

Proof. Let O be the centre of " S ". From the metrical properties of isocouples, OP coincides with OR, and if OP meets " S " in T, $OP \,.\, OR = OQ^2 = OT^2$.

Hence R is the middle point of the chord of contact of tangents from P to " S ". ....(α)

By Theorem I, Cor. 3, if the circular lines through Q meet " S " again in U, V, the chord UV passes through P.

Since the orthogonal pair of lines QX, QY separates (QA, QB) as well as (QU, QV) harmonically, (X, Y) are the double points of the involution on " S " to which (A, B), (U, V) belong. Since AB, UV meet in P, the tangents at X, Y also meet in P. ....(β)

From (α), (β) and Theorem I, Cor. (2), the results (1), (2), and (3) follow.

R being the Frégier-point of Q, QR is the normal at Q to " S ", and from what we have proved already, QR is also normal at Q to the circle QXY. Hence the circle QXY touches " S " at Q, and the common chords QX, QY of " S " and the circle are equally inclined to an axis of " S ".

§ 5. *The isocouple and the pedal conic of a quadrangle.*

$ABCD$ is a quadrangle of which the pairs of opposite sides are $T_1 \equiv (AB, CD)$, $T_2 \equiv (BC, DA)$, $T_3 \equiv (CA, DB)$, which intersect respectively in X, Y, Z. XYZ is the harmonic triangle of the quadrangle.

By Theorem III, the three line-pairs T_1, T_2, T_3 have a unique common isocouple (P, Q), which we call *the isocouple* of the quadrangle; and P is the centre of the rectangular hyperbola circumscribing the quadrangle. *

(P, Q) have the property that the lines joining them to any vertex of the harmonic triangle are equally inclined to the sides of the quadrangle which meet at that vertex and bear to each other a ratio equal to the cosine of the angle between these sides.

(P, Q) is an isocouple for all conics circumscribing the quadrangle and also for their pairs of asymptotes.

Let (l, l') be a pair of asymptotes of one of these conics. Then the perpendicular from P on l (or l') and that from Q on l' (or l) are in one-to-one correspondence, and the locus of their intersection is a conic through P and Q, which we may denote by Π. If the feet of the perpendiculars from Q on (l, l') be (H, K) respectively, the four points P, Q, H, K lie on Π, whose centre therefore is the common middle point of PQ and HK. The intersection of l and l' is evidently on the nine-point conic of the quadrangle $ABCD$, and is the orthocentre of the triangle PHK.

We may now condense the above facts in the following theorem:

Theorem XIV. If (P, Q) be *the isocouple* of a quadrangle, the locus of the feet of the perpendiculars from P on the asymptotes of any circumscribed conic of the quadrangle is a conic passing through eight points, viz. the feet of the perpendiculars on the six sides of the quadrangle and the isocouple (P, Q); PQ is a diameter of this conic, and if HK be any variable diameter, the orthocentre of the triangle PHK lies on the nine-point conic of the quadrangle.

N.B. We call this new conic *the eight-point conic* or *the pedal conic* of the quadrangle.

Cor. 1. The asymptotes of the eight-point conic are perpendicular to those of the nine-point conic; but the transverse axes of the two conics are in the same direction.

Cor. 2. The line joining the centres of the eight- and the nine-point conics is perpendicular to the tangent at P to the eight-point conic.

Cor. 3. The common points of the two conics lie on a circle whose diameter is PR, where R bisects PQ.

There are numerous other properties of this new conic which can be easily discovered by any interested reader from the equation given below; and we wish to conclude with the remark that the eight- and the nine-point conics are very intimately related to each

other. In fact, if we write the equation of the nine-point conic of the quadrangle $ABCD$ in Cartesian coordinates in the form

$$ax^2+by^2+2gx+2fy=0,$$

with $$P\equiv 0, 0, \quad \text{and} \quad Q\equiv \frac{4g}{b-a}, \frac{4f}{a-b},$$

taking for axes of reference the asymptotes of the rectangular hyperbola circumscribing $ABCD$, the equation of the eight-point conic becomes

$$(a-b)(bx^2+ay^2)+4(bgx-afy)=0,$$

showing that one may be derived from the other by a simple linear transformation of the type

$$X=\lambda x, \quad Y=\mu y.$$

A. A. K. A.

1062. Here is an interesting experiment that you can carry out with a napkin ring. Place it on your finger and whirl it round and round. . . . What keeps it above the table while it is moving rapidly is a force known as centrifugal force. The word means " flying from the centre ", but it is an unfortunate name, as it is not really flying from the centre at all, but at a tangent.—Children's page, *Daily Telegraph*, August 25, 1934.

1063. A MODEST OPENING.

There is nothing in the study of mathematics of greater importance than a thorough knowledge of algebraic factors. There are few subjects more difficult to master, especially if the student has not the advantage of a skilful teacher. When they are once mastered, the study of mathematics is interesting, and comparatively easy, but it is impossible for a student to make good progress in the higher branches of mathematics if he has not reached a high degree of proficiency in this part of the subject.—A. L. Sparkes, *Factors Simplified*, p. 1 (1879). (Just given to the Library by Mr. W. A. Gilmour.) [Per Prof. E. H. Neville.]

1064. AN UNCOMMON AMBIGUITY.

" The elementary ideas of the Calculus are not beyond the capacity of more than 40% of our Certificate candidates."

Does this assert that the ideas can be grasped by at least three-fifths of the class, or only that at least two-fifths can understand them ? [Per Prof. E. H. Neville.]

1065. TOMBSTONE MATHEMATICS.

There are queerer things on gravestones than the shorthand epitaphs mentioned in last Sunday's *Observer*. I believe the tomb of the mathematician, Leonhardt Van Euler, has engraved on it the evaluation of π (the relation of the radius to the circle) as far as the dead man had carried it. It runs to seventy figures (beginning, of course, with 3·14, etc.), and as he died in 1783 it is probable that the research has now been carried much further.—Letter in the *Observer*, March 8, 1936. [Per Prof. E. H. Neville.]

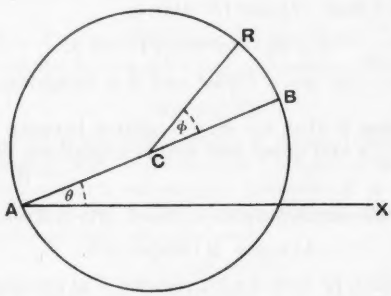
1066. Each of these, he found, had some two things to say about life, and three jokes, so that the conversation of each was a sort of recurring decimal of five places.—C. E. Montague, *Honours Easy*; included in *Great Short Stories of the War*, p. 841 (1930). [Per Capt. I. FitzRoy Jones.]

THE PILLORY.

London University Final Pass B.Sc. (Internal), 1921.

APPLIED MATHEMATICS. III.

"5. A smooth circular wire, of radius a, can turn freely in a horizontal plane about a vertical axis through a point on the circumference. A small ring of mass equal to that of the wire, initially at rest on the wire at the point diametrically opposite to the point through which the axis passes, is projected along the wire with velocity V. Find the angular velocity of the wire just before the ring reaches the axis."



Let A be the point through which the axis passes, C the centre of the wire, B the initial position of the ring, AX the initial position of AB. At time t, let AB have turned through angle θ and let the ring be at R, where $\angle RCB=\phi$.

Then the coordinates of R are

$$\{a(\cos\theta+\cos\overline{\theta+\phi}),\quad a(\sin\theta+\sin\overline{\theta+\phi})\},$$

and its resolved velocities are

$$-a\{\sin\theta\,.\,\dot\theta+\sin\overline{\theta+\phi}\,.\,(\dot\theta+\dot\phi)\},\quad a\{\cos\theta\,.\,\dot\theta+\cos\overline{\theta+\phi}\,.\,(\dot\theta+\dot\phi)\}.$$

The external forces have no moment about A. Therefore the angular momentum about A is unchanged.

Thus $\qquad mV.\,2a=2ma^2\dot\theta+ma^2(1+\cos\phi)(2\dot\theta+\dot\phi).$

Hence when $\phi=\pi$, $\dot\theta=V/a$.

This appears to be the solution expected by the examiner. But, we observe, the external forces do no work ; thus there is no change of kinetic energy ; that is,

$$\tfrac{1}{2}mV^2=\tfrac{1}{2}\,.\,2ma^2\dot\theta^2+\text{K.E. of ring.}$$

Hence if $\dot\theta = V/a$, the kinetic energy of the ring is negative. The complete energy equation is

$$a^2\{2\dot\theta^2(2+\cos\phi)+\dot\phi^2+2(1+\cos\phi)\dot\theta\dot\phi\}=V^2, \quad \ldots\ldots\ldots\ldots(A)$$

and the momentum equation is

$$a\{2\dot\theta(2+\cos\phi)+(1+\cos\phi)\dot\phi\}=2V. \quad \ldots\ldots\ldots\ldots\ldots(B)$$

Putting $\dot\phi=0$ in these we get

$$2a^2\dot\theta^2(2+\cos\phi)=V^2,$$

and $$a\dot\theta(2+\cos\phi)=V,$$

whence $\dot\theta=V/2a$ and $\phi=\pm\frac{1}{2}\pi$.

Eliminating $\dot\theta$ from (A) and (B) above,

$$a^2\dot\phi^2(3-\cos^2\phi)=2V^2\cos\phi.$$

Thus when $\phi=\pm\frac{1}{2}\pi$, $\ddot\phi=\mp V^2/3a^2$ and $\dot\phi$ is imaginary unless ϕ lies between $\pm\frac{1}{2}\pi$.

The conclusion is that the ring oscillates between the positions given by $\phi=\pm\frac{1}{2}\pi$ and never gets within a quadrant of the axis.

B. A. Swinden.

London Inter-Collegiate Scholarships Board. March Examination, 1935.

Applied Mathematics.

"8. A particle is projected with velocity V at an elevation θ to the horizontal, from a point on an inclined plane of slope α. Show that its range up the line of greatest slope of the plane is

$$\frac{2V^2\sin(\theta-\alpha)\cos\theta}{g\,.\cos^2\alpha}.$$

Show that the least velocity with which a stone must be projected from the ground in order to clear two parallel vertical walls of heights 14 ft. and 19 ft. at a distance 24 ft. apart is $8\sqrt{g}$."

"9. A car starting from rest and running freely down a slope of length 100 yds. and inclination 1 in 28 acquires a momentum sufficient to carry it a distance of 300 yds. along the level before coming to rest. If the car weighs one ton and can develop 12 h.p., what is the greatest speed it can attain up a slope of 1 in 16, and what will be its acceleration up this slope when its speed is 30 m.p.h.? The frictional resistances to motion are considered to be constant throughout."

A. S. Gosset Tanner.

Girton and Newnham Colleges.
Scholarship Examination in Natural Sciences, March 1933.

Advanced Physics. A.

"1. A body rolls without slipping down a smooth plane. Discuss its motion.

A sphere and cylinder of the same material and same radius keep pace with each other when rolling down the same inclined plane. What is the length of the cylinder in terms of its radius ?"

A body cannot *roll* down a smooth plane. The length of the cylinder has nothing to do with its pace. A. S. GOSSET TANNER.

Oxford Responsions, March 1936.

"Prove that two tangents to a circle from an external point are equal.

A quadrilateral has a circle inscribed in it. One pair of opposite sides of the quadrilateral are equal: prove that the other pair are parallel."

The second part is not true. K. S. SNELL.

Merton, New College, Brasenose, Christ Church and Hertford. Mathematical Scholarships and Exhibitions, March 1936.

MECHANICS.

"4. A mass M is dragged across a horizontal plane by an elastic horizontal string MP of unstretched length a and modulus λ, P moving with uniform velocity v. Initially M is at rest and the string slack. Show that, if the plane is smooth, M will over-run P when it has moved through a distance

$$2a + \pi v \sqrt{(a/\lambda)}.$$

If the plane is rough with coefficient of friction μ, show that the string never becomes slack, when $v^2 < a\mu^2 g^2/\lambda$; and that M never over-runs P, when $$v^2 < 2a\mu g + a\mu^2 g^2/\lambda."$$

In each result, for $1/\lambda$ read M/λ.

"7. Two smooth spheres of radii a, b and masses M, M' hang in contact, suspended from a smooth peg by a light string of length $2l$ whose ends are fastened one to each sphere at a point of its surface. Show . . . that the action between the two spheres at their point of contact is

$$\frac{(a+b)\sqrt{(MM')}}{\sqrt{\{l(a+b+l)\}}}."$$

The dimensions are incorrect, as a factor $\frac{1}{2}g$ is omitted.

K. S. SNELL.

London Higher School Certificate, 1929. *Group D.*

APPLIED MATHEMATICS. III.

"A number of small masses are placed at various points on a rough horizontal plate, which is caused to rotate about a vertical axis, the coefficient of friction between all the masses and the plate being the same. If the plate is originally at rest and moves with constant

angular acceleration, prove that the masses will start to slip at times which are inversely proportional to the square roots of their distances from the axis of rotation, these times being measured from the instant at which the plate starts to move."

Since the angular velocity is increasing, the transverse acceleration is $r\ddot{\theta}$ and the radial acceleration is $-r\dot{\theta}^2$.

Hence the resultant acceleration is $\sqrt{(r^2\ddot{\theta}^2+r^2\dot{\theta}^4)}$.

But $\qquad\qquad \ddot{\theta}=\lambda$, a constant.

Hence $\qquad\qquad \dot{\theta}=\lambda t$, since plate starts from rest.

Thus resultant acceleration is $\lambda r\sqrt{(1+\lambda^2t^4)}$.

Slipping will occur when

$$m\lambda r\sqrt{(1+\lambda^2t^4)}=\mu mg,$$

that is, when

$$t^4=\left\{\left(\frac{\mu g}{\lambda r}\right)^2-1\right\}\bigg/\lambda^2$$

$$=\frac{\mu^2g^2}{\lambda^4r^2}-\frac{1}{\lambda^2},$$

and

$$t=\left(\frac{\mu^2g^2}{\lambda^4r^2}-\frac{1}{\lambda^2}\right)^{\frac{1}{4}},$$

and is *not* proportional to $1/r^{\frac{1}{2}}$.

F. T. Chaffer.

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette.*

Sir,—It is better to let other people see you helping yourself to humble pie than to live in the expectation of having a large piece handed to you one day. The method of obtaining the group of partial fractions associated with a multiple quadratic factor in the denominator of a rational function which I described in 1922 in the *Gazette* (vol. XI, p. 10) and the *Messenger of Mathematics* (vol. LII, p. 39), was given in 1861 in the *Quarterly Journal* (vol. V, p. 39), by Joseph Horner; the article has the title "Decomposition of Rational Fractions", and is dated October, 1860. The writer remarks that the method of determining the fractions corresponding to $x-a$ in $f(x)/\{(x-a)^nF(x)\}$ by direct division of $f(a+y)$ by $F(a+y)$ has not become general "notwithstanding its decided arithmetical superiority to any other", and although it is now half a century since Chrystal advocated the method in the first volume of the treatise to which we all pretend to pay homage, the complaint is as well grounded as it ever was. Yours etc.,

E. H. Neville.

The University, Reading,
May 13, 1936.

MATHEMATICAL NOTES.

1198. *The parametric equation of the normal to a parabola.*

If we represent the parabola $y^2 = 4ax$ by the parametric equations $x = at^2$, $y = 2at$ then the condition that the normal at "t" shall pass through the point (α, β) is

$$at^3 + (2a - \alpha)t - \beta = 0.$$

If the point "t" is the vertex of the parabola and the point (α, β) is the cusp of the evolute, namely $(2a, 0)$, then this equation reduces to $t^3 = 0$, that is, it has three equal roots.

We may use this last fact to determine the focus and axis of the parabola given by the general parametric expressions :

$$x = at^2 + 2bt, \quad y = ct^2 + 2dt. \quad \text{......(i)}$$

For brevity write

$$\lambda = ab + cd, \quad \mu = ad - bc,$$
$$\nu = a^2 + c^2, \quad \rho = b^2 + d^2.$$

We assume $\mu \neq 0$, otherwise the locus given by (i) is a straight line.

The equation of the normal is

$$x(at + b) + y(ct + d) = t(\nu t^2 + 3\lambda t + 2\rho). \quad \text{......(ii)}$$

This passes through a point (α, β) if

$$\nu t^3 + 3\lambda t^2 + (2\rho - a\alpha - c\beta)t - b\alpha - d\beta = 0. \quad \text{......(iii)}$$

If "t" corresponds to the vertex (X, Y) and (α, β) is the cusp of the evolute, (iii) has all its roots equal and so

$$t = -\lambda/\nu, \quad \text{......(iv)}$$
$$2\rho - a\alpha - c\beta = 3\lambda^2/\nu, \quad \text{......(v)}$$
$$b\alpha + d\beta = -\lambda^3/\nu^2. \quad \text{......(vi)}$$

Substituting the value of t in (i) the coordinates of the vertex are given by
$$\nu^2 X = -\lambda(a\lambda - 2c\mu), \quad \nu^2 Y = -\lambda(c\lambda + 2a\mu), \quad \text{......(vii)}$$
and from (ii) the equation of the axis is, after removing a factor μ,
$$cx - ay = 2\lambda\mu/\nu. \quad \text{......(viii)}$$

If we solve equations (v) and (vi), we find

$$\nu^2\alpha = -a\lambda^2 + 2d\mu\nu, \quad \nu^2\beta = -c\lambda^2 - 2b\mu\nu. \quad \text{......(ix)}$$

The focus (x_1, y_1) is midway between (X, Y) and (α, β). Hence

$$\nu x_1 = d\mu - b\lambda, \quad \nu y_1 = -b\mu - d\lambda. \quad \text{......(x)}$$

The length $4p$ of the latus rectum is twice the distance from (X, Y) to (α, β). This gives at once

$$4p = \frac{4\mu\sqrt{\rho}}{\nu}. \quad \text{......(xi)}$$

The directrix is perhaps most easily found as the line through the intersection of the perpendicular tangents

$$x = -b^2/a \quad (\text{at } t = -b/a),$$

and

$$y = -d^2/c \quad (\text{at } t = -d/c),$$

at right angles to the axis (viii). The result is

$$ax + cy + \rho = 0. \qquad \text{.........................(xii)}$$

B. E. LAWRENCE.

1199. *A lesson on ratios.*

I feel that the *Gazette* would be more helpful to a large number of our members if it contained more articles on the teaching of elementary work and even notes of lessons. Here are my recollections of a lesson that may prove of interest.

The class had already discussed the ratios of the areas and volumes of similar figures, and had done some examples out of class which have been looked over and just returned.

Teacher. Can you have the ratio of 7 cm. to 6 ft. ?

Class (unanimously). No.

T. Why not ?

C. For a ratio the two quantities must be in the same unit.

T. Can you have the ratio of the length of this piece of chalk to the length of the blackboard ?

C. Yes.

T. This piece of chalk is 7 cm. long and the blackboard is 6 ft. long ; can you have the ratio of the two lengths ?

C. Yes.

T. Then is it incorrect to speak of the ratio of 7 cm. to 6 ft. ?

C. No. But it is not nice.

T. Now make a statement about ratios.

After some discussion the class evolved the following : You can have the ratio of two quantities of the same kind, and to express this *numerically* they must be measured in the same unit.

We then considered the following question which they had done out of school. (The question is shortened here.)

" Corresponding lengths in two similar figures measure 2 in. and 5 ft. and the area of the first figure is 14 sq. in. ; what is the area of the second ? "

Nearly all had solved it by working throughout in inches. I then asked what they would do if the given lengths had been 2 cm. and 5 ft. and the given area 14 sq. cm.

One boy suggested changing the centimetres to feet ; another suggested that we should assume 1 ft. $=k$ cm., and hope that the k's would drop out.

We then worked as follows :

Let A sq. ft. be the required area.

Let 1 ft. $=k$ cm.; therefore 1 sq. ft. $=k^2$ sq. cm.

Hence we found $$\frac{Ak^2}{14}=\left(\frac{5k}{2}\right)^2.$$

We then considered keeping the units in our ratios and worked as follows:

The ratio of the linear dimensions of the two figures $=\dfrac{5\text{ ft.}}{2\text{ cm.}}$.

Therefore the ratio of their areas $=\dfrac{25\text{ sq. ft.}}{4\text{ sq. cm.}}$.

Let A sq. ft. be the area of the second figure; we know that the area of the first is 14 sq. cm.

Thus the ratio of their areas is $\dfrac{A\text{ sq. ft.}}{14\text{ sq. cm.}}$.

Hence $$\frac{A\text{ sq. ft.}}{14\text{ sq. cm.}}=\frac{25\text{ sq. ft.}}{4\text{ sq. cm.}}.$$

Thus $$\frac{A}{14}=\frac{25}{4}.$$

A. W. S.

1200. *The circumscribed, inscribed and escribed circles of a triangle in trilinear coordinates.*

The method given in most textbooks for obtaining the equation of the circle circumscribing the triangle of reference involves a rather irritating question of positive and negative signs, while the deduction of the equation of the inscribed circle usually involves a certain amount of algebraic manipulation. It may be worth while, therefore, to give the following simple proofs.

1. The circumscribed circle.

Its equation is of the form

$$fyz+gzx+hxy=0,$$

and its centre is the point $(\cos A, \cos B, \cos C)$. The polar of the centre is

$$(g\cos C+h\cos B)x+(h\cos A+f\cos C)y+(f\cos B+g\cos A)z=0.$$

Identifying this with the line at infinity, viz.

$$ax+by+cz=0,$$

which can be written in the form

$$(b\cos C+c\cos B)x+(c\cos A+a\cos C)y+(a\cos B+b\cos A)z=0,$$

we have immediately $\quad f:g:h=a:b:c.$

2. The inscribed circle.

Its equation in line-coordinates is of the form

$$Fmn+Gnl+Hlm=0,$$

and the pole of the line at infinity is, in point-coordinates,

$$(Gc+Hb,\ Ha+Fc,\ Fb+Ga).$$

Identifying this with the centre, viz. (1, 1, 1), we have

$$Gc+Hb=Ha+Fc=Fb+Ga,$$

which give immediately, on solving,

$$F:G:H=a(s-a):b(s-b):c(s-c).$$

The equation in point-coordinates is

$$\sqrt{(Fx)}+\sqrt{(Gy)}+\sqrt{(Hz)}=0,$$

$$\text{i.e. } \sqrt{\{a(s-a)x\}}+\sqrt{\{b(s-b)y\}}+\sqrt{\{c(s-c)z\}}=0.$$

3. The escribed circle opposite the vertex A.

The centre now is $(-1, 1, 1)$, giving the equations

$$Gc+Hb=-Ha-Fc=-Fb-Ga,$$

which are the same as those for the inscribed circle except that $-F$, $-a$ take the places of F, a. Changing a into $-a$ changes s, $s-a$, $s-b$, $s-c$ into $s-a$, s, $-(s-c)$, $-(s-b)$, respectively, and so we get

$$-F:G:H=-as:-b(s-c):-c(s-b),$$

$$\text{i.e. } F:G:H=-as:b(s-c):c(s-b),$$

whence the point-equation of the escribed circle is

$$\sqrt{(-asx)}+\sqrt{\{b(s-c)y\}}+\sqrt{\{c(s-b)z\}}=0.$$

Similarly for the other two escribed circles.

4. Finally, a word about the circular points at infinity.

From the relations

$$c=a\cos B+b\cos A,\quad 0=a\sin B-b\sin A,$$

we have, on multiplying the second by $\pm i$ and adding, the important trigonometric identities

$$c=ae^{\pm iB}+be^{\mp iA},$$

which express that the two points $(e^{\pm iB}, e^{\mp iA}, -1)$ lie on the line at infinity.

Also, the equation of the circle circumscribing the triangle of reference becomes

$$ayz+bzx+(ae^{\pm iB}+be^{\mp iA})xy=0,$$

$$\text{i.e. } ay(z+xe^{\pm iB})+bx(x+ye^{\mp iA})=0,$$

which demonstrates the fact that it passes through the points $(e^{\pm iB}, e^{\mp iA}, -1)$. These are, therefore, the circular points.

H. A. Hayden.

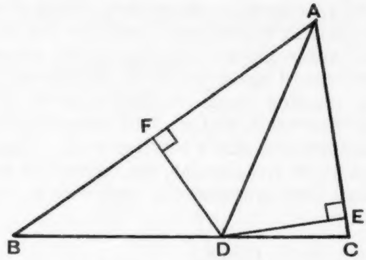
1201. *A proof of a theorem in elementary geometry.*

I recently set a " B " School Certificate form to prove the following theorem, and one boy produced a proof not as elegant as the usual

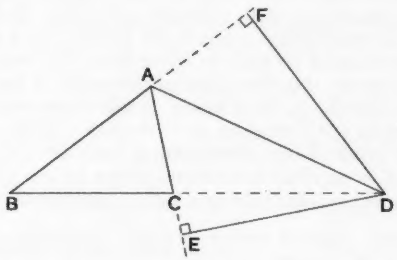
one but somewhat shorter and decidedly simpler in construction. It may not be new, but I have never seen it before.

To prove that the bisector of an angle of a triangle divides the opposite side in the ratio of the other two sides.

Given a triangle *ABC* with the angle at *A* bisected, internally or externally, by *AD*, which cuts *BC*, or *BC* produced at *D*.



In each figure, draw *DE* and *DF* perpendicular to *AC* and *AB* respectively.



As *AD* bisects the angle at *A*, it is the locus of points equidistant from *AB* and *AC*.

Therefore $$ED = FD.$$

Therefore $$\frac{\triangle ABD}{\triangle ACD} = \frac{AB}{AC}.$$

Also, as triangles *ABD* and *ACD* each have sides common to *BC*, and the third vertex in common,

$$\frac{\triangle ABD}{\triangle ACD} = \frac{BD}{DC}.$$

Therefore $$\frac{AB}{AC} = \frac{BD}{DC}.$$

It will be noticed that the converse theorem follows simply, by merely reversing the argument. S. TRUSTRAM.

1202. *The Rôle of i in the Special Theory of Relativity.*

Sir James Jeans has said that in relativity we measure time in terms of a mysterious unit which involves i. Why ? Because this weird unit enables us to express the results of the theory of relativity in the simplest possible form (*The Mysterious Universe*, p. 110). " If we are further asked why this is so, we can give no answer." The ultimate why is no doubt a problem for the metaphysician, but the question as to why i occurs in our representation of certain concrete and observable facts in experimental physics can be answered.

Consider the " space plane " diagram which represents time and the single dimension of space to which the special theory confines its attention in the first instance. Let ot, ox be the time and the space axes of an observer S, and ot', ox' those of a second observer S', moving with uniform velocity v relative to S. Then, assuming the two sets of axes to be rectangular, we expect the coordinates of an event in the space-time systems of S' and S to be connected by the relations :

$$\left.\begin{aligned} x' &= x\cos\theta - t\sin\theta, \\ t' &= x\sin\theta + t\cos\theta, \end{aligned}\right\} \text{ where } \theta = \text{arc}\tan v. \quad \ldots\ldots\ldots\ldots(a)$$

But our expectations are belied by the paradoxical behaviour of light—by the constancy of its velocity—which the Michelson-Morley experiment has established. Since the velocity of a particle determines the direction of its path in space-time, the constancy of light velocity must imply the isotropism of the paths of light rays. Thus *we are led to identify the light tracks in space-time with the isotropic (circular) lines in our Euclidean representation.* That is to say, lines which on the basis of our experimental data are $x = \pm ct$, can only be represented in an ideal Euclidean scheme by $x = \pm it$. If we wish to bring our ideal scheme into line with experimental data, we must change the unit of time in such a way that t becomes $\pm \frac{c}{i} t$, and hence $v = \frac{x}{t}$ becomes $\frac{x}{\pm \frac{c}{i} t} = \pm \frac{vi}{c}$. Changing t and t' thus in (a), we get the well-known transformation formulae of Lorentz. We thus see that i is bound to enter into our formulae when the paradoxical behaviour of light is fitted into the Euclidean representation of space-time.

The paradox about light is also a paradox about time ; for we see that the unit of time employed in experiment masks the essential characteristic of light. The unit of time which accords with the strange behaviour of light in space-time (conceived as an Euclidean space scheme) is not a second, but $\pm \frac{c}{i} \times$ one second. This conclusion invites comparison with a theory which Mr. J. W. Dunne has recently put forward in his book, *The Serial Universe.* Mr. Dunne assumes that time has a " regressive " character and shows (*ibid.*, p. 145) that when this character is fitted into the spatial represen-

tation of time which experimental physics presupposes, the recorded unit of time has to be multiplied by ki where k is " the velocity of the 'now' " (the upper bound of the velocity of any material body).

C. T. RAJAGOPAL and C. T. KRISHNAMA CHARI.

1203. *The differential operator* $\left(x\dfrac{d}{dx}\right)^{-1}$.

A student of mine asked me to explain the following discrepancy.

If m is a positive integer, then

$$\left(x\frac{d}{dx}\right)^m x^\alpha = \alpha^m x^\alpha, \qquad \text{(i)}$$

whereas apparently

$$\left(x\frac{d}{dx}\right)^{-m} x^\alpha = \frac{1}{\left(x\dfrac{d}{dx}\right)^m} x^\alpha$$

$$= (\alpha+1)^{-m} \,.\, x^\alpha, \qquad \text{(ii)}$$

the interpretation of $\left(\dfrac{d}{dx}\right)^{-1}$ or $1\Big/\left(\dfrac{d}{dx}\right)$ being taken as an integration with respect to x. There is evidently a discrepancy between equations (i) and (ii). If now we write $x=e^z$, we have

$$\left(x\frac{d}{dx}\right)^{-m} x^\alpha = \left(\frac{d}{dz}\right)^{-m} e^{\alpha z},$$

which being interpreted to mean integration with respect to z, m times, gives us

$$\left(x\frac{d}{dx}\right)^{-m} x^\alpha = \alpha^{-m} x^\alpha, \qquad \text{(iii)}$$

a result in complete agreement with (i).

The explanation is as follows. If, as is usual, we interpret

$$\left(\frac{d}{dx}\right)^{-1} \text{ or } 1\Big/\left(\frac{d}{dx}\right)$$

to mean an integration with respect to x, we must interpret the operator $\left(x\dfrac{d}{dx}\right)^{-1}$ in the manner indicated by the following equation

$$\left(x\frac{d}{dx}\right)^{-1} f(x) = \frac{1}{\dfrac{d}{dx}} \cdot \frac{f(x)}{x}, \qquad \text{(iv)}$$

and *not* by the equation

$$\left(x\frac{d}{dx}\right)^{-1} f(x) = \frac{1}{x\dfrac{d}{dx}} \cdot f(x). \qquad \text{(v)}$$

To prove this, on writing $x=e^z$, we have

$$\left(x\frac{d}{dx}\right)^{-1} f(x)=\left(\frac{d}{dz}\right)^{-1} f(e^z)$$

$$=\int f(e^z)\,dz$$

$$=\int f(x)\,\frac{dx}{x}$$

$$=\frac{1}{\frac{d}{dx}}\cdot\frac{f(x)}{x}.$$

With this interpretation it is easily seen that

$$\left(x\frac{d}{dx}\right)^{-m}.\,x^a=\frac{1}{\frac{d}{dx}}\cdot\frac{1}{x}\left[\frac{1}{\frac{d}{dx}}\cdot\frac{1}{x}\cdots\frac{1}{\frac{d}{dx}}\cdot\frac{1}{x}\right]x^a$$

$$=\alpha^{-m}.\,x^a$$

in complete agreement with equation (i).

From this point of view equation (iv) is a symbolic way of writing

$$\int f(x)\,\frac{dx}{x}=\int f(e^z)\,dz$$

where $x=e^z$. C. Fox.

1204. *The* m*th differential coefficient of* $\sin^n x$.

If $$\cos x+i\sin x=y,$$
we know that

$$2\cos kx=y^k+y^{-k},$$

$$2i\sin kx=y^k-y^{-k}.$$

Thus $$2^n i^n \sin^n x=(y-1/y)^n$$

$$=[y^n+(-1/y)^n]+\frac{(-i)\,n}{1\,!}[y^{n-2}+(-1/y)^{n-2}]$$

$$+\frac{1}{2\,!}(-i)^2 n(n-1)[y^{n-4}+(-1/y)^{n-4}]+\ldots .$$

If n is even,

$$2^{n-1}i^n\sin^n x=\sum_{s=0}^{\frac{1}{2}n}\frac{(-)^s}{s\,!}\,n(n-1)\ldots(n-s+1)\cos(n-2s)\,x\,;$$

and if n is odd,

$$2^{n-1}i^{n-1}\sin^n x=\sum_{s=0}^{\frac{1}{2}(n-1)}\frac{(-)^s}{s\,!}\,n(n-1)\ldots(n-s+1)\sin(n-2s)\,x.$$

But, if $y=\sin ax,\quad y_m=a^m\sin(ax+\tfrac{1}{2}m\pi)$,

and if $y=\cos ax,\quad y_m=a^m\cos(ax+\tfrac{1}{2}m\pi)$.

Thus, if n is any integer,

$$\frac{d^m}{dx^m}(\sin^{2n}x)=\frac{i^{2n}}{2^{2n-m-1}}\sum_{s=0}^{n}\frac{(-)^s}{s!}\,2n(2n-1)\ldots$$
$$(2n-s+1)(n-s)^m\cos\{2(n-s)x+\tfrac{1}{2}m\pi\},$$

and

$$\frac{d^m}{dx^m}(\sin^{2n+1}x)=\frac{i^{2n}}{2^{2n}}\sum_{s=0}^{n}\frac{(-)^s}{s!}(2n+1)(2n)\ldots$$
$$(2n-s+2)(2n-2s+1)^m\sin\{(2n-2s+1)x+\tfrac{1}{2}m\pi\}.$$

A. F. MACKENZIE.

1205. *Solution of* $u=a+e\sin u$ *in series.*

If $$u=a+e\sin u,$$
by Lagrange's theorem we have

$$u=a+\sum_{n=1}^{\infty}\frac{e^n}{n!}\frac{d^{n-1}}{da^{n-1}}(\sin^n a),$$

a result due to Laplace (Edwards, *Differential Calculus*, p. 455, No. 6). Using the series for $\frac{d^m}{da^m}(\sin^n a)$ given in the preceding note, we have

$$u=a+\sum_{n=1}^{\infty}\Bigg[\frac{e^{2n}i^{2n}}{2n!}\sum_{s=0}^{n}\frac{(-)^s}{s!}\,2n(2n-1)\ldots$$
$$(2n-s+1)(n-s)^{2n-1}\cos\{2(n-s)a+\tfrac{1}{2}(2n-1)\pi\}\Bigg]$$
$$+\sum_{n=0}^{\infty}\Bigg[\frac{e^{2n+1}i^{2n}}{(2n+1)!\,2^{2n}}\sum_{s=0}^{n}\frac{(-)^s}{s!}(2n+1)(2n)\ldots$$
$$(2n-s+2)(2n-2s+1)^{2n}\sin\{(2n-2s+1)a+n\pi\}\Bigg].$$

A. F. MACKENZIE.

1206. *A simple transformation.*

In a previous note*, the author has exhibited a simple transformation which changes an equation of the form $y=ax^n$ into a straight line $y=a(X-1)$. He has also pointed out that data suspected of conforming to the power law can thereby be tested without the use of logarithms, and incidentally, the value of a can be determined by either the slope or y-intercept of the resultant straight line.

The application of this transformation, with slight modification, can be generalized.

If the given equation is

$$y=ax^n+bx^{n-1},$$

then the application of the transformation

$$y=Y(X-1)^{n-1},$$
$$x=X-1$$

* *National Mathematics Magazine*, Nov. 1934, p. 53.

causes the equation to become

$$Y = a(X-1) + b,$$

from which we can find a and b.

To illustrate the idea, let the given data be

x - -	1	2	3	4	5
y - -	-2	-4	0	16	50

Suppose that we suspect these data of conforming to the law

$$y = ax^3 + bx^2.$$

Then the equations of the transformation would be

$$y = Y(X-1)^2 = Yx^2,$$
$$x = X - 1,$$

which changes $y = ax^3 + bx^2$ into

$$Y = a(X-1) + b.$$

Solving the equations of the transformation for Y and X, we find

$$Y = y/x^2.$$
$$X = x + 1.$$

Therefore, to find the new X's and Y's corresponding to the original x's and y's we merely add 1 to each given x to obtain the new X and divide the old y by the square of the corresponding x to obtain the new Y.

TRANSFORMED DATA.

X - -	2	3	4	5	6
Y - -	-2	-1	0	1	2

Since these data when plotted can be seen to conform to a straight line, the original data must have conformed to the law

$$y = ax^3 + bx^2.$$

The value of a is the slope of the straight line, and b is equal to the y-intercept of the straight line plus the value of a.

In general, an equation of the type

$$y = ax^n + bx^{n-1} + \ldots + tx^{n-r},$$

can be changed into $\quad Y = a(X-1)^r + \ldots + t$

by the transformation $\quad y = Y(X-1)^{n-r},$

$$x = X - 1.$$

L. J. ADAMS.

CORRIGENDA.

"The rehabilitation of differentials" (Vol. XX, No. 238).

P. 121. Ex. 2 should read "η^2 is infinitesimal compared with η".

P. 122. Line 4 should read "$a + c = b + d$".

P. 131. Line 1. For "jusitfy" read "justify".

REVIEWS.

David Hilbert. Gesammelte Abhandlungen. III. Analysis. Grundlagen der Mathematik. Physik. Verschiedenes nebst einer Lebensgeschichte. Pp. vii, 435. Inlandpreis RM. 45, Auslandpreis RM. 33.75. 1936. (Springer)

This is the third and final volume of Hilbert's collected works. It contains or deals with his contributions to analysis, the foundations of mathematics and to mathematical physics. It includes also his paper on Mathematical Problems, his obituary notices of Minkowski, Hurwitz, Weierstrass, Darboux, and a life history of Hilbert by Blumenthal.

The paper of Hilbert's of most interest to the general reader is undoubtedly the famous one on Mathematical Problems delivered at the International Mathematical Congress at Paris in 1900. The subject has always been a favourite one, but I would describe his as easily the foremost and most significant and fruitful of all such papers. It contains twenty-three problems drawn from the most different branches of mathematics, *e.g.* foundations, geometry, algebra, arithmetic, groups, analysis. Not all of them are new, but most of them reflect his great and active interest in and knowledge of wide domains of mathematics.

It commences as follows: " Which of us would not willingly lift the veil under which the future lies concealed in order to throw a glance upon the impending progress of our science and the secrets of its development during the coming century ? What will be the particular goals that the leading mathematical spirits of the coming generation will strive to attain ? What new methods and new achievements will the new century reveal in the wide and rich field of mathematical thought ? "

A partial answer to this is given by the subsequent history of the problems. An account of the progress made and of the developments arising out of the problems, a large number of which have been completely solved, was given by Bieberbach in 1930. Many of them have supplied the germ of new theories directing and shaping the future course of mathematics, and even the last year has seen the brilliant work of Gelfond and Schneider upon transcendental numbers and of Siegel on quadratic forms and the modular functions of n variables.

The obituary notices of Minkowski and Hurwitz are of peculiar interest. They will be appreciated the more by reading Blumenthal's most interesting and fascinating account of Hilbert's life history. Hilbert was a native of Koenigsberg and was born there in 1862; while Minkowski came there in 1872 as a boy of eight. Minkowski was a youthful genius and had before he was eighteen produced his brilliant memoir on quadratic forms which shared with that of H. J. S. Smith the prize offered by the French Academy. Minkowski spent five semesters at the University of Koenigsberg and then went to Berlin where he listened to Kummer, Kronecker, Weierstrass and Helmholtz. Hurwitz, who was three years older than Hilbert, and who was a student and protégé of Klein's, came to Koenigsberg as extraordinarius in Easter, 1884, while Hilbert was a student. Hilbert was soon drawn into scientific intercourse with Hurwitz, and Minkowski, who spent his vacations at Koenigsberg, joined the circle. There soon developed the closest ties of friendship, cooperation and emulation, not to mention an eight-years' study of every part of mathematics. Hilbert does full justice to both of them, and makes full acknowledgment of his debt to both of them as a source of knowledge and inspiration.

This volume is particularly rich in examples of collaboration and association between Hilbert and " non-Aryan " mathematicians. In addition to Hurwitz, Minkowski and Blumenthal already mentioned, there are Courant, Cohn-

Vossen, Bernays and Hellinger. Certainly this has been to the great advantage of mathematics. The results do not lend support to Bieberbach's recent arguments on racial types among mathematicians.

The papers on analysis deal chiefly with the Principle of Dirichlet, the Calculus of Variations and Integral Equations. The collected works do not reproduce Hilbert's contributions to the latter subject. Instead, an appreciation is given by Hellinger who points out that Hilbert's work arose from his efforts to include, by a simple general line of approach, the greatest possible domain of linear problems of analysis, and in particular the linear boundary problems of ordinary and partial differential equations and the expansions arising therefrom. Hilbert's aim was always that of finding methods based upon a broad foundation and leading to further results, an aim which he had achieved in his work on Dirichlet's Principle.

This is a method of argument which Dirichlet—induced by an idea of Gauss—had applied to the solution of the problem of finding a potential function $f(x, y)$ with assigned values on a given boundary curve. Draw perpendiculars to the x, y plane at the points P of the boundary curve equal to the given boundary value at P. Their extremities form a curve in space which may be considered as the boundary of an infinity of surfaces. Let $z=f(x, y)$ be that one for which

$$J(f)=\iint\left\{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right\}dx\,dy$$

is a minimum. Such a surface is easily shown to be a potential surface by the calculus of variations. It was by considerations of this kind that Riemann considered the existence of a solution of the boundary problem as settled and then without hesitation based upon this his magnificent theory of Abelian functions. It was Weierstrass, however, who was the first to point out, what would now be obvious to most first year Honours students, that a minimum need not exist. This meant that Dirichlet's proof was not valid and so for many years it was merely of historic value, though this might not be evident from the emphasis placed upon it in some of our textbooks in use about the beginning of the century.

The existence of a minimum, however, was established by the important investigations of Neumann, Schwarz and Poincaré, who proved the solvability of the boundary problem under very general conditions. It seemed most unfortunate, however, that the attractive simplicity of the fundamental idea of Dirichlet's Principle, with its undeniable abundance of possible applications to pure and applied mathematics and its inherent suggestiveness should be abandoned. It remained for Hilbert to bring the long dead principle to life again and to introduce the necessary ideas to make it a rigorous method of fundamental importance. Never was there a more apt illustration of Klein's remark in his lecture on Riemann: "It is by the reconsideration of old problems with new methods that pure mathematics grows".

It is not difficult to see that Hilbert's views should have led him to make a study of the foundations of various branches of mathematics. It has been a popular belief that rigour in a proof was the enemy of simplicity. But as Hilbert says in his Mathematical Problems, this is a mistaken idea. He asserts that on the contrary there are numerous examples where the rigorous methods are simpler and more easily grasped. The effort towards rigour forces us to find simple methods of argument. Frequently it facilitates the way to methods that are more pregnant than the older ones of less rigour. It is not surprising then that at least nine papers, of which one is an account by Bernays of Hilbert's work on the foundations of arithmetic, in the present volume, are concerned with foundations.

I cannot refrain from quoting a few remarks by Klein, who was a colleague of Hilbert's and was responsible for bringing him to Göttingen, expressing a rather different point of view upon the occasion of his lecture on Riemann:

"Certainly it is the final object in the construction of any mathematical theory to produce a rigorous proof for all assertions. Certainly mathematics itself pronounces judgment when it refrains from rigorous proofs. But the secret of gifted productivity will always remain that of ever finding new questions, of foreseeing new theories from which open out important results and relations. Without the creation of new points of view, without the setting up of new aims, mathematics would soon exhaust itself in the severity of its logical methods of proof and begin to stagnate because of a possible lack of material. Hence in a certain sense mathematics has been advanced most by those who are distinguished more for intuition than for rigorous methods of proof."

I need only conclude by remarking that Hilbert cannot be called a voluminous writer compared with many other mathematicians, even when note is taken of his contributions published as separate volumes and not included in his collected works. But there is no other in recent years whose work has covered so extensive a range, or for whom new and general ideas, and not involved calculations or myriads of details, played so important a *rôle*, or whose discoveries have been so fundamental and so epoch-making as his. There will not be many mathematicians from whose collected works so much pleasure and profit will be had either from a brief examination or a close study.

L. J. MORDELL.

General Analysis. I. By E. H. MOORE. Pp. vi, 231. 13s. 6d. 1935. Memoirs of the American Philosophical Society, Vol. I. (American Philosophical Society, Philadelphia; Oxford Press)

This book is the first of a series of four volumes intended to give an account of E. H. Moore's *General Analysis*, in particular of its later developments since 1915. Moore's own papers on this subject are only fragmentary and not available to a large mathematical public. Therefore one has to be thankful to the American Philosophical Society for having undertaken this publication. The revision of Moore's notes and their preparation for print was carried out by R. W. Barnard, who has made valuable contributions to the theory himself and is therefore very competent for this task.

In 1906 Moore laid down his well-known principle of generalization by abstraction:

"The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to these central features."

From this point of view he considered the manifold analogies between the theory of linear equations in a finite number of unknowns, the theory of the Fredholm integral equation and Hilbert's theory of linear equations in infinitely many variables. He founded an abstract theory which comprised all three of them as special instances (First General Analysis Theory, 1906-1915).

Further progress in the theory of integral equations and of infinitely many variables (Hilbert, Hellinger, Radon) made it necessary for him to modify his theory in order to include these new results (Second General Analysis Theory, 1915-1932).

Consider the integral equation

$$(1) \qquad \int_a^b K(x, y)\phi(y)\,dy = f(x),$$

where K and f are given functions and ϕ is the unknown function. To generalize it in a wider sense one has to define

(i) a set of objects generalizing the functions K, f, ϕ,

(ii) a functional $\mathfrak{J}$ that coordinates a number to each pair of "functions". $\mathfrak{J}$ is used to generalize the left-hand side of (1).

In his second theory Moore proceeds as follows :

All numbers that occur belong to a given field $\mathfrak{A}$. $\mathfrak{A}$ is either the field of all real numbers or that of all complex numbers or, finally, that of all real quaternions. In $\mathfrak{A}$ all usual postulates about number fields are valid, except the commutative law of multiplication which need not be satisfied. Every number x of $\mathfrak{A}$ has a conjugate $\bar{x}$, and we put $|x|^2 = x\bar{x}$.

The values of every function are supposed to belong to $\mathfrak{A}$. The arguments of our functions range over the elements of a given abstract aggregate $\mathfrak{P}$. p, p', p'' denote general elements of $\mathfrak{P}$; $p_1, p_2, \ldots, p_n$ any selection of a finite number of different elements of $\mathfrak{P}$. Put $\sigma = \{p_1, p_2, \ldots, p_n\}$.

We are further given a function $\epsilon(p', p'')$. The "matrix" $(\epsilon(p', p''))$ is supposed to be Hermitian and positive definite, *i.e.* for every σ, $\xi(p)$ the form

$$F_\sigma(\xi) = \sum_1^n \overline{\xi(p_i)}\sigma(p_i, p_k)\xi(p_k)$$

is Hermitian and positive definite, if one considers the $\xi(p_i)$ as independent variables.

Now we introduce an absolute value of "vectors" $\xi(p)$ by putting

$$M\xi = \text{upper bound of } \left| \sum_1^n \overline{\alpha(p_i)}\xi(p_i) \right|$$

for all σ, α satisfying $\quad F_\sigma(\alpha) \leqslant 1.$

Let $\quad (\gamma_\sigma(p_i, p_k))_{i,\, k=1, 2, \ldots, n}$

be the inverse of the matrix

$$(\epsilon(p_i, p_k))_{i,\, k=1, 2, \ldots, n}.$$

Then for every pair of vectors $\xi(p)$, $\eta(p)$, we put

$$\mathfrak{J}_\sigma \bar{\xi}\eta = \sum_1^n \overline{\xi(p_i)}\gamma_\sigma(p_i, p_k)\xi(p_k),$$

$$\mathfrak{J}\bar{\xi}\eta = \lim_\sigma \mathfrak{J}_\sigma \bar{\xi}\eta, \tag{2}$$

provided the limit in (2) exists. This limit is taken in the following sense. To every real number $e > 0$ there corresponds a σ_0, such that

$$|\mathfrak{J}_\sigma\bar{\xi}\eta - \mathfrak{J}\bar{\xi}\eta| < e,$$

for all σ satisfying $\sigma_0 \subset \sigma$.

In particular this limit always exists if $M\xi < \infty$, $M\eta < \infty$. We have

$$\mathfrak{J}\bar{\xi}\xi = |M\xi|^2$$

and the Schwarz inequality $\quad |\mathfrak{J}\bar{\xi}\eta| \leqslant M\xi \,.\, M\eta.$

Now we can state the generalization of (1). Let $\kappa(p', p'')$ and $\alpha(p)$ be given functions. We ask for a function $\xi(p)$ such that

$$\mathfrak{J}_{p''}\overline{\kappa(p', p'')}\xi(p'') = \alpha(p').$$

Here $\mathfrak{J}_{p''}$ is the operator $\mathfrak{J}$, working on the variable p''.

Of the problems belonging to this field I will only mention here the characteristic value problem and the generalized Fourier theory.

The present book, Part I of the whole series, deals with the algebra of finite matrices whose elements belong to a field $\mathfrak{A}$ of the type described above.

Chapter I gives fundamental facts about matrices, such as are connected with the notion of rank and reciprocal of a matrix. Since multiplication is not necessarily commutative, there is no equivalent to the notion of determinants, and the proofs have to use methods different from those usually employed. They are valid in every field, satisfying all usual field postulates except possibly $ab=ba$.

In Chapter II fields $\mathfrak{A}$ are considered where to every element x a conjugate $\bar{x}$ is defined, and where this correspondence has the usual properties. In fact all such fields are closely related to one of the three special fields mentioned above.

Here an interesting definition of the determinant of an Hermitian matrix is given which generalizes the usual definition in the commutative case and retains some of its fundamental properties. Furthermore, it is shown that every Hermitian form can be written in the form

$$\sum_{k=1}^{n} \alpha_k \,|\, \beta_{k1}\xi_1 + \dots + \beta_{kn}\xi_n \,|^2.$$

In Chapter III an order relation is introduced in $\mathfrak{A}$, generalizing the relation $\leqslant$. Then it becomes possible to introduce the notion of the signature of an Hermitian form and to prove some theorems generalizing well-known results in the classical case.

In stating theorems and definitions, the book makes ample use of logic symbols. This, in most of the cases, contributes to the clearness of the style. A detailed list of all symbols and permanent notations used in the book proves to be very convenient. R. R.

An Introduction to the Theory of Elasticity for Engineers and Physicists. By R. V. Southwell. Pp. viii, 509. 30s. 1936. Oxford Engineering Science Series. (Oxford)

A treatise on Elasticity by the Professor of Engineering Science at Oxford must in any case attract a very much wider public than merely engineers and physicists, in view of the high reputation of Professor Southwell as a mathematician. Indeed even the professional mathematician will find some stiff reading in the book, especially when he gets to the later chapters dealing with vibrations and elastic stability, of which subjects the author is a recognized master.

There are really two books here, which have been compressed into a single volume. The first, which extends to the end of Chapter VII, is obviously intended as a textbook for engineering students with the object of giving them, in a simple form, those fundamental results in the Theory of Elasticity which they require to know, and it attempts to restrict itself to the mathematical equipment which they may be expected to have. This, however, soon gets exhausted and we have, so to speak, a new deal. In Chapter VIII we get to grips with the proper theory and after this, except that special prominence is given to engineering applications, the main appeal must be to mathematical specialists.

Clearly the large majority of readers will be more interested in the first part, and here we note the ever-present danger in this class of work that, in trying to make mathematical arguments intelligible to the engineer, one may succeed in making them unintelligible to the mathematician, a danger which is accen-

tuated, in the present case, by the order which the author has adopted. For at the start he eschews all consideration of what is usually understood by stress and strain and bases his treatment entirely on external forces and deflections, linearly related by a " Hooke's law " which is essentially identical with the Principle of Superposition. This, of course, necessitates that deflections must be small, but this fundamental assumption is nowhere made clear. For example, Kirchoff's uniqueness theorem is given in § 14, but the unsuspecting reader is never warned that the theorem actually fails in elastic solids if the displacements are large.

Even if displacements are small, simple cases can be constructed where they are not proportional to the load, as indeed is very well pointed out in § 21. One cannot help feeling, however, that, after reading this, the confidence of the beginner will receive a severe shock; it is only much later that he will discover that the really important things are strains, not displacements.

Again, it seems a little unfortunate to introduce strain-energy before strain has been either defined or mentioned; thus we find a proof of De Saint Venant's Principle in § 93 which involves a distribution of strain-energy in various portions of a beam, but we have to wait until § 115 before we reach the concept of strain-energy in a volume. Incidentally, it appears doubtful whether the proof in question would be passed by most mathematicians as valid.

Another trouble is that we constantly have either important qualifications omitted, or future developments and results anticipated, with an occasional appearance of contradiction. One example (out of many) occurs in § 181, when a " total shearing action " F suddenly appears in a discussion on flexure of beams which uses formulae established for flexure without shear. No explanation of the contradiction is given until we reach § 223, forty-eight pages later.

One rather wonders whether this habit of " making things easy for the engineer "—a habit by no means restricted to the work under review—by suppressing what are very real mathematical difficulties, does in fact help the engineering student, and whether it is not at the bottom of much of the invincible repugnance and mistrust with which most practical engineers regard mathematics. Is it really, for instance, more difficult for a student to master the perfectly simple and clear classical exposition of the transformation of stress given in § 272 (simplified, it may be, by restriction to two dimensions) than the complicated figures and lengthy calculations of §§ 130-135 ?

The second half of the book is naturally free from the above objections and one feels that here the author has given himself a free hand. Most of it is excellent, and one's main regret is that Professor Southwell has had to cut down the space allotted to some of the more interesting methods. The discussion of the " transmitted " solutions for flexure and torsion is admirable, the treatment of the rectangular section by means of the approximate form of § 352 being particularly neat.

The problems treated in this half are too many and varied to enumerate and they include practically all those which are of real importance to the engineer.

Inevitably there are some criticisms of detail: thus, there seems perhaps an exaggerated tendency to overstress recently published " methods " of somewhat special interest, to the detriment of work of classical importance. One is astonished, for example, to find Clerk Maxwell's method of reciprocal figures for stresses in simple frameworks dismissed in two lines and described as " Bow's notation ", with a casual and doubtful reference to Clerk Maxwell in a footnote; especially in view of the fact that Maxwell's method is practically of universal application and will, *inter alia*, deal easily with the framework shown in Fig. 33, without the need for the " method of sections " or the " method of interchange ".

In discussing the conditions of compatibility of strain, one could have wished that the author had gone a little further and given the "consistency equations" for the stresses, at any rate for statical problems under no body-force, in the usual form

$$(1+\sigma)\nabla^2\widehat{pq}+\frac{\partial^2}{\partial p\,\partial q}(\widehat{xx}+\widehat{yy}+\widehat{zz})=0, \quad (p, q=x, y, z)$$

which enable the stresses to be solved for directly.

It would also have been helpful, in view of the importance attached to "self-straining" in the earlier part of the book, to have elucidated this somewhat difficult concept in its most general form and to have shown its relation to the compatibility conditions for strain.

The account of the photo-elastic method in § 411 misses entirely the significance of the isoclinic lines (which are not even mentioned) and would lead one to imagine that the isochromatic lines alone, whether obtained with white or with monochromatic light, will suffice for stress exploration. One would also like to know what theoretical or experimental justification there is for the suggestion in the footnote to p. 374 that it seems on theoretical grounds more probable that the retardation is determined by strain rather than by stress.

Again, it is not easy to see the need for the investigation of § 439, which relates the stress-functions for plane strain and plane stress when there is body-force. For it does not, in fact, relate stress-functions for identical problems because the boundary conditions are different in the two cases, so that an independent calculation is still necessary; and it leaves the reader unaware that, in the important case where $\nabla_1^2(\rho\Omega)=0$, the two stress-functions for identical problems can be related, which is what is really wanted, and that χ' is then equal to the mean value of χ taken across the thickness, there being no factor $(1-\sigma)^2/(1-2\sigma)$ such as appears in equation (66).

There is one oversight which should be corrected in a future edition: on p. 129 we are told that the angular acceleration of a block about an axis is equal to the moment of the external forces about that axis, divided by the corresponding moment of inertia, a statement which contradicts Euler's equations unless the block is a cube.

As regards the vexed question of notation, in the latter half of the book the author follows Love and in the earlier half the notations more commonly used by engineers. He has not adopted Pearson's notation $\widehat{pq}$, although he admits its advantages when dealing with curvilinear coordinates. There seems no valid reason for this distinction, and, if Pearson's notation is adopted, it sets free the symbols X_x, etc., for use to denote "stress-resultants" in the theory of plates.

One cannot help regretting that an opportunity has been lost of familiarizing the engineering student from the start with a logically consistent and self-explanatory notation. However, much may be forgiven Professor Southwell for having had the courage to adopt the word "slide" to denote what most engineers habitually term shear or shearing strain. For him to have gone the whole distance and dropped the clumsy "shear stress" in favour of "shear" would have been sheer temerity.

After all is said, however, the fact remains that anything Professor Southwell writes on this subject is imbued with the characteristic vitality of the pioneer and that his book is a storehouse of precious information which no scientific engineer and no mathematical physicist can afford to neglect.

L. N. G. Filon.

Essays on Examinations. Pp. xii, 168. 5s. 1936. International Institute Examinations Enquiry. (Macmillan)

To those responsible for the conduct of examinations, the limitations and imperfections of examinations are only too familiar. To them this collection of essays, interesting as it is, will, it may be feared, bring little help. To the outside world in so far as it is disposed to attach excessive value to examination results, it will perhaps be more useful.

That the existing system does not accomplish all that is hoped for even by the better informed is true enough ; that it is capable of improvement is also true—and indeed improvement is constantly in progress.

Whether, as some of the essayists seem to hold, an alternative system is possible which will in fact accomplish the aim of putting every peg into its proper hole is more doubtful—perhaps neither the pegs nor the holes are sufficiently definite, or constant, in shape.

Sir M. Sadler contributes an interesting essay on the English scholarship system, taking the term in a very wide sense. His salient points are : (1) in England the State has never yet taken a dominating position in regard to education ; (2) scholarships, in one form or another, have provided the links which have made tolerable inequalities which in the absence of such links would have led to explosions ; (3) competition, introduced into education generally by the Jesuits, found its way into the scholarship system ; (4) admission to the Civil Service came to be dependent on academic competition. So that, in short, success in examinations of one kind or another has become to very many people the means by which they have obtained their start in life.

This being so, it is clearly of vital importance that examinations should be both " reliable " and " valid "—technical terms which Professor Spearman in his short note explains. If different examiners, or the same examiners at different times, produce widely discordant marks for the same papers the results are " unreliable ". If they agree, the results may, so far, be reliable—but not necessarily valid. For " validity " has reference not to the mere conduct of the examination, but to its purpose. Until it is understood what an examination is intended to achieve, or in Professor Spearman's language, to measure, no judgment can be formed as to its validity. In practice the trouble too often is that the actual or original purpose is ignored and values are attached to examination results which they will not properly carry.

Dr. Ballard writes on the Special Place examinations. Here the proper purpose of the examination is fairly clear, namely, to find out which children of eleven are fit for one or other of certain forms of higher education. The dominating factor, it is agreed, should be the native intelligence of the child. Accordingly the results of the usual examinations can fittingly be compared with those arrived at by " intelligence tests ".

As to this, Professor Burt states (p. 102), " Where teaching is efficient, where attendance is regular and where the curriculum is fairly uniform throughout the area, there the ordinary examination . . . proves to be fairly efficient. But in heterogeneous areas . . . it appears that a large number of bright children may easily fail in an examination that is purely scholastic ". Dr. Ballard has much severer things to say. " These examinations flagrantly fail to satisfy the conditions of validity and reliability " ; but he admits that improvement is going on. The mischief is intensified in some areas at least by the fact that the examination is competitive whereas its proper purpose is qualifying. But inasmuch as there are various forms of higher education this difficulty will remain in one form or another if the examination is to " aim at appraising, not merely general ability, but those specific aptitudes and interests

which should determine the true place of each pupil in the post-primary system ".

Dr. Burns writes on the " Social Needs of the Modern World " giving "some indication of the social problems which should be considered when the examination system is criticised ". The weakness of his article is that he overlooks the proper relation between examination and curriculum, assuming that examinations dominate education, whereas, in theory at least and to a large extent in practice, the nature of examinations is dependent on the nature of the education. He distinguishes between " examinations which are mainly tests of further educability " and " those which are regarded as ways of exit from the educational world to the world outside it ". Among the latter he includes " school leaving examinations ", that is, the School and Higher School Certificate examinations. And this shows at once that his primary classification is faulty. For it means that he views, *e.g.* the School Certificate examinations as " tests of competence on entry into the world of occupations ". " They are not suitable tests for those intending to be railwaymen or textile workers ". The last sentence quoted is of course ambiguous. If it means that possession of the School Certificate is no adequate proof that the holder will be successful as a railwayman or as a textile worker, it is a truism. If it means that the future railwayman is the worse for having a School Certificate, it is very questionable doctrine, educationally and socially.

In either case it has no bearing on the " validity " of the examination ; for the avowed purpose of the examination is not to ascertain whether a person is fitted for a particular career but the much humbler one of ascertaining whether or not he has pursued a prescribed course of study with reasonable success. If employers individually or collectively attach values to the certificate which it does not properly bear, the error is theirs. Dr. Burns makes the same mistake in respect of University examinations. " In practice ", he says, " our universities are producing men and women of adequate competence ; but their systems of examination are not serving as tests of the kind of competence most valuable in modern conditions ". The immediate answer is the same as before —the examinations are intended to test certain courses of study ; what their value as indications for the future may be is, strictly speaking, irrelevant.

On the whole Dr. Burns' article is really an indictment, whether justifiable or not, not of the examinations, but of the education of which the examinations form a necessary part.

Mr. Abbott, dealing with a more narrowly defined field than Dr. Burns, is able to show how, under certain conditions, that which Dr. Burns desires is, with limitations, being done. Incidentally his article illustrates the mode of action of the State in this country—cooperation not domination.

He tells how a proper scheme of professional education leading to examinations for certificates has been developed for each of certain well-defined professions. On the other hand he says, with truth, " the system of national certification is not capable, without complete modification, of extension to branches of industry where Institutions of this standing and character do not exist ". And it must be remembered that these schemes, even where they are possible, are concerned with persons who are already committed to the relevant industry—they are intra-professional, not general.

The article most directly relevant to existing school and examination conditions is that of Sir P. Hartog on " English composition at the School Certificate examination ". It is based on the finding : " The Investigators have found much evidence that at present certificates may be granted to candidates whose English is lamentably weak ". Sir Philip is perplexed at the inability or the unwillingness of the Investigators to give a " blunt and simple affirmative

in reply " to their own question " Should a reasonable command of English be required as a condition of obtaining a certificate ? " In his perplexity, searching about for reasons, he lights on one which is grotesquely absurd though he " sees no other explanation possible ".

His own article goes some, though not the whole, way to justify the rooted distrust some people feel towards much of the current teaching and examination in " composition ". He quotes some subjects set in recent examinations and comments: " To some persons, though probably not to all, such subjects will seem ludicrous, for the essay test ought to be a test of the power of the average boy or girl to write clearly and systematically on a given subject with a definite object in view—the kind of power which a business firm might reasonably demand from its employees ".

Sir Philip lays stress on the need for writing for a definite audience ; he might well have made more explicit the need for a definite subject-matter. One of the examples he gives of the consequences of failure to write properly is that of men with first-class honours degrees being discharged by the Department of Scientific and Industrial Research because they were unable to write reports intelligible to manufacturers. The real vice of the ordinary essay-subject is the lack of definite content known to the writer. He has to feel about for something to say—anything, to fill a page or two, whether he believes or cares about what he says or not. The real problem for most people is to give a clear exposition of material which is familiar to them ; training and examination in composition which do not keep this aspect of the subject in the foreground miss the main point and are on the whole contemptible.

Not the least interesting articles are the three on the Leaving Examinations in Prussia. The first, a memorandum of Mr. Sadler's, has for long been familiar to students of education, but is well worth reprinting ; and it gains in interest from the two which follow on the examinations before and after the Nazi Revolution.

W. C. F.

The Mathematical Theory of Finance. By KENNETH P. WILLIAMS. Pp. xii, 280. 12s. 1935. (Macmillan Co., New York)

The title of this book might lead one to expect a work covering a much wider range than that with which we are presented. The mathematics required does not go beyond that given in any elementary algebra ; the highest point being the Binomial Theorem. The low estimate which the author takes of his possible reader is shown by the following advice : " The student is urged to work the following exercise : From the equation

$$A = P(1 + ti)$$

express each of the three quantities P, t, and i in terms of the other three letters in the equation ".

It would have been thought that a student whose algebra was in such a primitive state was better advised to defer the study of the Mathematical Theory of Finance until his mathematics was in a less precarious condition.

For a student who knows his algebra well, as far as the Binomial Theorem, the work under review forms a good introduction to the simpler interest problems which occur in investments, given in the notation agreed upon at the Second International Actuarial Congress of 1898 and now generally accepted.

The headings of the various chapters are: Simple Interest and Simple Discount, Compound Interest, Equation of Value and Equation of Payment, Annuities, Different Conversion and Payment Periods, Sinking Funds and Amortization, Bonds, Depreciation and Replacement, Life Annuities, Life Insurance and Appendix on the Binomial Theorem, and Arithmetic and Geometric Progressions, Logarithms.

Pages 206 to 273 are given up to tables showing the values of $(1+i)^n$, v^n $s_{\overline{n}|}$, $a_{\overline{n}|}$, $a_{\overline{n}|}^{-1}$ at various rates of interest for integral values of n up to $n=50$ and then by 10-year intervals to 100, also tables of the logarithms of the same functions and a general 6-place logarithm table.

The author has presented the subject clearly, and gives good advice to the student against the too mechanical use of formulae. He seems, however, to use occasionally the method against which he warns the student; thus on p. 77 under the captions *The annuity that will amount to* 1, and *The annuity that* 1 *will buy*, he refers back to (4) Chapter IV, and (9) Chapter IV; we will quote the latter. "If in (9) Chapter IV we put $A=1$, we see that the periodic payment of the annuity is $1/a_{\overline{n}|}$. *Thus* $1/a_{\overline{n}|}$ *is the annuity that* 1 *will buy*."

Now the formula referred to (9), Chapter IV, is

$$A = Ra_{\overline{n}|},$$

which formula has been derived from first principles; $a_{\overline{n}|}$ is defined as representing the present value of 1 for n annual payments, and surely we have from the Golden Rule $a_n : 1 :: 1 : 1/a_{\overline{n}|}$ without reference to any formula containing concrete units.

There are a few terms, peculiar to U.S.A., which we should be sorry to see introduced in this country. Thus for $s_{\overline{n}|}$, the author says "read 's angle n' ". If a pronounceable name is required, it is a pity to use the accidental shape of the symbol as a name; far better that the name should indicate the nature of the function. Where we say, "if the interval (of the argument) is wider", he says, "if the 'bracket' is wider"; if a bond decreases uniformly he speaks of it as "the straight-line method".

It is not clear what the author means by his use of the word "logical". Thus on p. 3, he says that "simple interest does not have a logical foundation"; and on p. 21, speaking of Compound Interest, he says "It is therefore logically superior".

But surely if the formulae for S.I. and C.I. are correctly derived, they must equally follow the strict laws of logic. Simple Interest is, however, the more usual form, Compound Interest being really applicable only in the case of very large and numerous transactions issued by the larger Companies who can do something approximating to the reinvestment of the S.I. when it falls due. There are many people who still seem to think that, though "men may come and men may go", interest goes on for ever. They should know better by now, and remember that a single penny invested at 5% for 2000 years (only a small portion of mankind's life on the earth) will amount to such a sum that the whole of the inhabitants of the earth could not pay it even if each of them possessed a globe of gold the same size as the earth. Yet it is often ignored that both interest and capital are variables. The Greeks were well aware of this, and that C.I. can only be used for short terms. This can be seen in the "Clouds" by Aristophanes. It is hard to speak too highly of the great aid that mathematics has been in questions regarding finance; the author, however, seems to have succeeded in doing so, when he says in his preface "Mathematics has given us life annuities and life insurance". "These benefits could not have come into existence without algebra."

This is quite contrary to historical fact, and it would be safe to say that Life Insurance Companies would have existed and carried on business of the simpler forms pretty much the same as at present (though of course not with the same security) if they had restricted themselves to arithmetic only. This book may very well serve as an introduction to the subject, before the actuarial student takes up his work in earnest.

W. S.

A Treatise on Hydromechanics. Part II. Hydrodynamics. By A. S. Ramsey. Fourth edition. Pp. xii, 415. 16s. 1935. (Bell)

A fourth edition seldom receives much more from the reviewer than a mention and appraisement of any changes from the third, but although Ramsey's book first appeared in 1912 as the long-awaited second part of Besant's *Treatise on Hydromechanics*, no review of the work has yet appeared in the *Gazette*.

The first edition dates from the time when the " classical " hydrodynamics appeared to be capable of little further development, and when it was regarded (albeit apologetically) by the very mathematicians who had lavished upon it so much ingenuity as an abstraction which (unfortunately) failed to explain some of the more evident properties of real fluids in motion. The subject had nevertheless manifest attractions, on account of the elegance of some of the methods used, and of the way in which it exemplified some fundamental dynamical principles and theorems. It was also a fruitful subject from the point of view of a Tripos examiner. Vestigial remains of this era are still to be seen in most chapters of the book, and especially in the collections of problems at the end of each chapter. These exercises, however, form a very valuable feature, giving as they do the embryo mathematician scope for the exercise of his craft so necessary to his development, denied to him in the more monumental treatises such as the classic of Lamb. Nor can it be doubted that these exercises, combined with the fact that the book is based on lectures given to Cambridge students, explain in a large measure why three editions have been exhausted in a quarter of a century.

With the rise of aviation, and the enormous amount of theoretical and experimental research into fluid flow which this rise has evoked in the last thirty years, the classical hydrodynamics of perfect fluids has proved to be capable of giving a much more complete account of some of the more important problems encountered than was believed possible when the first edition appeared. To-day the need of an initial mastery of the " classical " methods, and especially of those appropriate to two-dimensional motion, is recognized even by those whose interest is entirely practical. English students are therefore fortunate in the existence of a textbook in which these methods are developed with care and clarity as in Ramsey's book. That the book has moved with the times is clear, if only from the new matter contained in the latest edition, such as the use of the powerful methods of contour integration for the calculation of the resultant forces and moments due to fluid pressure in two-dimensional motion, and the entirely new chapter upon the motion of viscous fluids. The emphasis of the book is still definitely mathematical rather than physical ; it is significant, for instance, that neither the Venturi meter nor the Pitot tube receives mention ! Apart from the opening chapter and the later one on sound waves, the assumption of incompressibility is implicit throughout. One might have expected more reference to the effects of compressibility, even if only to the fact that speeds must rise to a quite appreciable fraction of that of sound before these effects have any practical significance. One also feels that this assumption of incompressibility should have been kept more explicitly in evidence. Possibly the author's use of the word " liquid " is regarded by him as sufficient—but one is shaken, and driven to enquire in what other cases the results cease to hold for compressible fluids, by the very occasional references (such as on p. 218), to our " regarding the fluid as incompressible ".

The new chapter on the motion of viscous fluids is a very valuable addition, and forms an excellent introduction to this difficult and complicated subject. The Navier-Stokes equations are established (and written out in full when

transformed to cylindrical and spherical polar coordinates, for which we put our gratitude on record !), and some important exact solutions, including a sketch of von Kármán's treatment of the rotating disc, are given. The solutions of Stokes and Oseen for the small motion of a sphere are included, followed by an introduction to Prandtl's " Boundary layer " theory as far as the Blasius solution for the flat plate, and a short mention of the problem of turbulence. On p. 373, in the first of equations (4), q_r^2 should be replaced by q_θ^2.

It is evident that this book will be even more useful to the present and succeeding generations of students than previous editions have evidently been in the past. W. G. B.

Tables of the Higher Mathematical Functions. I. By H. T. Davis. Pp. xiii, 377. 25s. 1933. (Principia Press, Bloomington, Indiana)

Professor Davis is a new entrant into the field of extensive table-making. He has undertaken an ambitious programme that will fill six or more volumes, and, with the aid of twenty collaborators, now presents his first volume.* It includes a historical section, a description of the analysis that precedes computation (misleadingly called " instruments " of calculation), a chapter on interpolation, and a bibliography of tables of the higher mathematical functions. The tables given are for interpolation, and of the gamma and psi functions.

With an unknown author, it is desirable to check considerable portions of the tables, to see if he has the reliability of Andoyer and Peters, the proneness to error of Steinhauser, Gifford and Hayashi, or the plagiaristic tendencies of Duffield, Benson and Ives. To this end 2000 values were specially computed, by interpolating to tenths 12-figure values in the *British Association Mathematical Tables*, Volume I. This checked Tables 2 and 8, containing 10-figure values of $\Gamma(x)$ and $\psi(x)$ for $x=1{\cdot}000(0{\cdot}001)2{\cdot}000$. The first table was obtained (see p. 192) by taking antilogarithms (using Vega) of the 12-figure logarithmic values given by Legendre. The result of the examination was:

C – D tenth decimal	No. of values
0	422
+1	209
–1	283
+2	29
–2	56
–3	1 (at 1·986)
+4	1 (at 1·564)

The error at 1·986 appears to be due to a slip ; that at 1·564 is due to an error in Legendre's logarithm, the last three figures of which should be 446, not 346. This error in Legendre should have been detected in the proof-reading, which should have included the building up of the function from the third differences. Again, differencing of $\Gamma(x)$ would have revealed this error. Apart from these two instances, the existence and distribution of the discrepancies corresponds to expectation, based on the process used.

Table 8 was computed (see p. 289) by sub-tabulation from the original table of Gauss, and checked by differencing. The value for 1·017 should end in 79, not 80. But far more serious is the fact that in the nine values from 1·431 to 1·439 the sixth decimal is too small by 1. The explanation of this is to be found in the method of sub-tabulation used (see p. 87), namely the well-known method of interpolating each interval separately, by computing as many leading differences as are necessary (with extra decimals as desired), and then

* A second volume appeared at the end of 1935.

building up from a constant higher-order difference. The great weakness of this method is that an error in passing from the first difference to the function may easily remain undetected if the computer is careless in his comparison of the original and the built-up values of the function at the end of the interval. In the present case an error in the sixth decimal was evidently made when forming $\psi(1{\cdot}431)$, but was not detected (as it should have been) when adding the last first difference to $\psi(1{\cdot}439)$ to form $\psi(1{\cdot}440)$. Smaller errors at 1·927–1·929 would appear to arise from the same cause; the tenth decimal has been made 1 too large at 1·927, and this has not been detected when forming $\psi(1{\cdot}930)$. The gross errors at 1·431–1·439 have also been missed when the table was " checked by differencing ", and also in the proof-reading, where, as we have already remarked, an experienced reader would build up.

From the values of $\psi(x)$ thus computed, 10-figure logarithms of $\psi(x)$ were formed. These were certainly not differenced (as they should have been), so that a fourth opportunity was missed. An error also occurs in the logarithmic value at 1·794, which should end in 59585, not 49585.

Two other faults came to light. $\psi(x)$ passes through zero, so that the 10-decimal values given have from seven to ten significant figures. Yet 10-figure logarithms of these rounded-off values are given, although more significant figures were available from the sub-tabulation; thus some of the values given may be in error by hundreds of units in the last decimal. Again, Vega has been interpolated linearly, second differences being ignored; the error from this cause may (and does) attain five units of the last decimal. We might also remark that Vega is known to contain 303 last-figure errors; Peters' *Zehnstellige Logarithmen* is correct, and, because of its better typography, is much easier to use.

The general lay-out of the tables, we regret to say, shows a lack of acquaintance with many of the elementary principles of tabulation, lack of consistency, and lack of consideration for the user. Thus:

(1) Rounded-off differences have been given in places, thus increasing the probable error of interpolates, mainly by the rounding off of the first difference.

(2) Forward differences have been given on the same line as the function—a relic of the days when Newton's forward-difference formula was used. Thus the second difference is a line too high for those who use central-difference formulae. Odd differences (if given) should be printed interlinearly.

(3) Whole columns of third differences of one significant figure are given, although with Bessel's formula third differences are negligible up to 40 or 50, and with Everett's formula completely negligible. Similarly, unnecessary fourth, fifth and sixth differences have been given.

(4) There has not been any systematic policy about using ciphers before decimals—sometimes they are in, sometimes they are not.

(5) Vertical rules have been used excessively; spaces are better between a function and its differences.

(6) Blocks of five lines begin with numbers of the form $5n+1$, instead of the conventional $5n$. To secure this, there are many blocks of six lines.

(7) In several tables blocks of ten lines have been used—for no apparent reason.

(8) In some tables the last line is repeated (as it should be) at the top of the next page; in others it is not—again for no apparent reason.

(9) Leading figures and signs have been repeated excessively, thus giving the page a very solid appearance.

(10) Equal-height (modern-face) figures have been used. These are inferior, for tables, to the old-style head and tail figures.

(11) In Table 7 the function is divided (by a space) between the fifth and sixth decimals, but the first difference between the sixth and seventh. On pp. 122-123 the function is divided after the sixth decimal and the differences after the fifth ; this is criminal.

(12) On pp. 122-123 the first difference retains its useless preliminary ciphers, while the higher differences drop theirs.

The above list, which could easily be doubled, indicates sufficiently the need for more editorial care.

The bibliography, in common with all others known to the writer, omits Newman's table of e^x in Volume XIV of the *Transactions of the Cambridge Philosophical Society*.

The chapter on interpolation puts forward methods and formulae that belong to the nineteenth century. Newton's, Stirling's and Gauss's formulae have given way to those of Bessel and Everett. The only method of inverse interpolation mentioned (and laboriously illustrated) is that of reversed series, which no practical computer would use. When the results of two calculations that should agree differ by 44 units in the last decimal, the author calmly drops two decimals instead of ascertaining the cause of the discrepancy ; there are three errors :

(1) The leading term of the last algebraical coefficient, namely

$$2-3m-5m^2+15m^3-14m^5+6m^6$$

should be 1, not 2.

(2) A large number of significant terms, although given in algebraical form, have been omitted in the calculation.

(3) The two terms with coefficients C and D are 10 times too large.

When allowance is made for these, the discrepancy is reduced to 1 in the last decimal.

The method of sub-tabulation described (p. 87) is also obsolete. Sang's masterly condemnation * of its use by Prony in the *Tables du Cadastre* may be read with profit by all addicted to its use. The errors in $\psi(x)$ are excellent testimony to its weakness.

The printing throughout of even-order differences, as in Table 9, would have saved many pages ; these differences suffice with modern methods of interpolation. This saving could have been enhanced by the use of the throw-back, which in turn would lead to further saving on the part of the user.

In spite of all the above criticism, which is intended to be helpful and constructive, and the inartistic printing, table-lovers are assured that they should possess this work. But we appeal for more consideration for the user in future volumes. L. J. C.

Funktionentafeln, mit Formeln und Kurven (Tables of Functions, with Formulae and Curves). By E. Jahnke and F. Emde. Second (revised) edition. Pp. xviii, 333. 16 marks. 1933. (Teubner)

Jahnke and Emde, as it is familiarly called, needs no introduction to a large group of mathematicians and physicists, as the first edition of 1909 and its two reprints have been part of their working tools. This revised edition, doubled in size, will undoubtedly double its circle of friends.

* *Proceedings of the Royal Society of Edinburgh*, VIII, 421 (1874).

The purpose of the book is the presentation, to about 4-figure accuracy, of the higher mathematical functions, and the working formulae required in their application. Graphical representation, sometimes in lieu of tabulation, has been freely used. We find the sine, cosine and logarithmic integrals, the error function, theta functions and elliptic integrals and functions, Legendre functions, Bessel functions and the Riemann Zeta function. The greater part of these have been taken from published tables, the sources being freely acknowledged.

A novel feature is the introduction of " reliefs ", representing the general character of complex functions by surfaces of which the ordinates are the moduli of the function. Contour lines of constant modulus and lines of greatest slope, *i.e.* of constant amplitude, give many of the reliefs a " chimney-stack " appearance. Of their value as a pictorial representation of the properties of a function there can be no doubt, and Emde's appeal to writers of textbooks to follow his example will not be unheeded.

The text is in German and English. Each section is followed by references to more extended sources of information ; we must protest, however, at the heading " More accurate tables " to lists containing tables by Hayashi, an author whose lamentable shortcomings have been dealt with in these pages and elsewhere. The omission of any reference to Peters' Calculating Tables (products up to 9999×99), or to Barlow's Tables, and the inclusion of tables of quarter-squares, are surprising.

It is unfortunate that English and continental notation for Bessel functions should not be in harmony. An English reader must study closely the definitions and formulae before he can identify the Y, I, K, ber and bei functions. On the other hand, the reviewer regrets the following of the English innovation of negative characteristics for the logarithms of numbers less than 1.

The printing is marred by the use of heavy condensed equal-height figures; Professor Emde could not do better than study the typography of the tables by his distinguished fellow-countryman, Professor Peters.

We read with pleasure a graceful tribute to the work of the British Association Mathematical Tables Committee (p. x), saying that " the mathematicians, physicists, and engineers of the whole world regard with the greatest wonderment and gratitude this colossal undertaking of their English colleagues, who have taken upon themselves almost entirely the heavy load of new computation ". L. J. C.

Mathematics for Technical and Vocational Schools. By S. Slade and L. Margolis. 2nd edition. Pp. xiii, 517. 12s. 6d. 1936. (John Wiley and Sons, New York ; Chapman and Hall)

This book was first published in 1922 and has now reached a second edition. The first half is devoted to the simplest arithmetic, algebra, geometry and trigonometry ; the second to calculations connected with machinery. The algebra is limited to the simplest simple equations, the geometry to the most elementary geometrical constructions, and the trigonometry to the solution of triangles by the sine and cosine rules, which are stated without proof.

The purpose and interest of the book are to be found in the examples of machine-shop calculations, designed for junior technical and trade schools, in the second part, under the following chapter headings, beginning with Chapter XV : strength of materials, work and power, woodwork, tapers, speed ratios of pulleys and gears, screw threads, cutting speed and feed, gears, milling-machine work, belting. The printing and figures are good, and the examples clearly set out. F. B.

Graphical Solutions. By C. O. Mackey. Pp. iv, 130. 12s. 6d. 1936. (John Wiley and Sons, New York; Chapman and Hall)

The first ninety pages of this book are devoted to mechanical and graphical means of representing the relation between the variables in a formula so that, when the values of all but one of them are given, that of the remaining one can be quickly obtained without further calculation. The three chief means described are slide-rules, net-work or intersection charts, and alignment charts. All these are useful to the engineer and receive some notice in elementary books on practical mathematics; here they are discussed at somewhat greater length; thus, slide-rules with more than one sliding scale, and alignment charts with one or more curved scales, are referred to and illustrated.

The rest of the book is concerned with finding empirical formulae to fit experimental data, when it is possible to anticipate formulae that can somehow be reduced to linear forms. Here again the treatment goes a little beyond its most elementary stage. For example, it is pointed out that, in the case of the parabola $y=a+bx+cx^2$, there is a linear relation between Δy and x if Δx is constant; again, if the experimental data can be expected to fit a law of the form $y=a+bx^n$, the value of a can first be found by choosing three convenient values of x in geometric progression, measuring the corresponding ordinates y_1, y_2, y_3, and calculating a from the equation $a=(y_1y_3-y_2^2)/(y_1+y_3-2y_2)$. The constants b and n can then be determined after writing the law in the linear form $\log(y-a)=\log b+n\log x$, by drawing a suitable straight line.

The book is well produced and numerous exercises are given. F. B.

School Arithmetic. Book I. 6th edition. Pp. 232. 3s. 10d. 1933. **Book II.** Pp. 258. 4s. 4d. 1928. **School Algebra. Book I.** Pp. 338. 5s. 4d. 1931. By M. H. Jurdak. (American Mission Press, Beirut)

These books, written for use in Syrian schools, by Mansur Hanna Jurdak, Professor of Mathematics in the University of Beirut, are not without interest to English teachers.

The present reviewer used to believe that the method of instruction in the Near East was mainly conning by rote. This belief received a rude shock some years ago when two boys educated in a Cairo lycée presented themselves for entrance at Dulwich College. Their work in elementary calculus showed a logical grasp and sense of style that must have been cultivated by first-rate teaching. In the books under review, which cover approximately the syllabus of School Certificate elementary mathematics, the instruction is in complete harmony with the recommendations of the Reports of our Association. It may be that Western influence accounts for this. Certainly Professor Jurdak is a member of the Association and of similar American societies. And it is worthy of notice that in the arithmetic of proportion the idea of ratio is more strongly emphasized than in some English textbooks.

The arithmetic includes the use of linear equations for the solution of problems. Many exercises are headed " Written exercise. Answer all you can orally ". There are also Time Tests; *e.g.* four long addition sums (tots) in $2\frac{1}{4}$ minutes.

The matter of the *Arithmetic* has a strange appearance. The book opens with the metric system and specific gravity. Local measures, Turkish and Cyprus measures and currency are included along with English. Four pages (22-26) are devoted to time and longitude, and pp. 158-168 are devoted to bills (not bills of exchange) and accounts. In fact a strong practical bias is evident.

F. C. B.

Cours de Géométrie. II. Géométrie dans l'Espace. By H. Estève and H. Mitault. Pp. viii, 283. 20 fr. 1936. (Gauthier-Villars)

This is the fourth volume of a series of books on geometry of which the first three volumes have already been reviewed in the *Gazette*. This volume is for the " classe de première ", average age sixteen.

It will perhaps suffice to give some indication of the contents of this volume and to say that those who found the first three volumes interesting will—if their interest in geometry is not entirely bounded by two dimensions—find this fourth volume equally interesting and provocative of thought.

As in an earlier volume, the material here is separated into Linear and Metric geometry, and this separation, by its postponement to Chapter V of perpendicularity of line and plane, seems perhaps even stranger here than before.

Book I, *Linear Geometry*, contains :

Chap. I. Straight line, plane, parallelism.

Chap. II. Linear transformations in space : (i) translation ; (ii) projection ; (iii) point symmetry ; (iv) homothety.

Chap. III. Surfaces polyhedral, cylindrical and conical (including harmonic pencils of planes).

Chap. IV. Volumes from the linear point of view. Prism, pyramid, cylinder, cone, frusta of pyramid and cone.

Book II, *Metric Geometry*, contains :

Chap. V. Straight line and plane ; perpendicularity (including dihedrals and symmetries of various kinds).

Chap. VI. Orthogonal projection (including the theorem of three perpendiculars, the shortest distance between two lines, the projection of areas including that of circle into ellipse).

Chap. VII. Surfaces of revolution, cylinder, cone, sphere.

Chap. VIII. Trihedrals and polyhedral angles from the metric point of view.

Chap. IX. Surfaces and volumes in metric geometry (this includes eighteen pages on spheres).

Book III, *Vectorial Geometry*, contains only

Chap. X. Elements of vectorial geometry in space, which includes the scalar product of two vectors but does not deal with their vector product.

The book ends with 203 exercises distributed fairly evenly at about twenty for each chapter. Each chapter also ends with a short set of questions of the essay type on book-work, " questions de cours limitées ".

There is no doubt that in addition to giving the English reader big questions to think about, such as whether it is well to separate linear from metric geometry, the perusal of such a book as this introduces him to many interesting oddments such as :

Is it possible to devise a proof more entirely unlike that of the English textbooks for the theorem that a line perpendicular to two lines in a plane is perpendicular to the plane ?

If our pupils replaced " obviously " by " visibly " should we use the blue pencil quite so vigorously ?

Are the pupils whom we castigate for spelling symmetry with one " m " merely showing themselves to be better French scholars than we are ?

If our pupils—but such questions are unsuitable in a serious review of what is undoubtedly a valuable and thought-provoking book. C. O. T.

Integralgeometrie. By W. Blaschke. Pp. 24. 7 fr. 1935. Actualités scientifiques et industrielles, 252; exposés de géométrie, I. (Hermann, Paris)

" Integralgeometrie " is the name Professor Blaschke suggests for a branch of mathematics known to English readers mainly by a connection with Crofton and Sylvester, the theory of geometric probability. In this little booklet the distinguished German geometer makes an initial contribution to a profound logical study of this theory by investigating certain invariant properties of linear sub-spaces of euclidean n-space. T. A. A. B.

Les théorèmes de la moyenne pour les polynômes. By J. Favard. Pp. 51. 15 fr. 1936. Actualités scientifiques et industrielles, 302; exposés sur la théorie des fonctions, I. (Hermann, Paris)

Rolle's theorem is essentially a theorem about a differentiable function of a real variable; we cannot expect, therefore, an extension in the real domain or an analogue in the complex domain unless we are prepared to place fairly severe restrictions on the function concerned. Professor Favard deals with polynomials; in the first part of his tract he takes Grace's beautiful (and somewhat neglected) theorem on the zeros of $f'(z)$ when $f(z)$ is a complex polynomial such that $f(a)=f(b)$, proves it by a neat application of Montel's method of " normal families ", and so leads us on through a succession of elegant results. In the second section he discusses real polynomials and links his theory with the moment problem of Stieltjes, with approximate integration and with polynomial approximation.

Incidentally, is it not time that the spelling of the name given as Tchebitchef by Professor Favard was standardized? T. A. A. B.

Séries lacunaires. By S. Mandelbrojt. Pp. 40. 12 fr. 1936. Actualités scientifiques et industrielles, 305; exposés sur la théorie des fonctions, II. (Hermann, Paris)

A lacunary power series is of the type

$$\Sigma a_n z^{\lambda_n},$$

where the integer exponents are such that the difference $\lambda_{n+1}-\lambda_n$ tends to infinity; the idea that this characteristic has a connection with the distribution of the singularities of the function defined by the series can be traced back to Weierstrass, who showed that if b is an integer greater than unity, the series

$$\Sigma a^n z^{b^n}$$

cannot be " continued " across the circle of convergence. But the first systematic researches are due to Hadamard, and have been continued by Ostrowski, Pólya and the author of this tract, among others. M. Mandelbrojt enunciates thirty-eight theorems, some of which deal with similar phenomena for Dirichlet series; he proves the more important and gives references for the rest, thus providing in small compass an outline of a fascinating branch of function theory. T. A. A. B.

A Mathematician Explains. By M. I. Logsdon. Pp. xii, 175. 7s. 1935. (University of Chicago Press; Cambridge)

Of the five uses suggested for this book in its preface the two most important are:

(i) To provide the mathematics for general physical science courses.

(ii) To serve as an eye-opener for the adult who knows no mathematics beyond elementary algebra and geometry but who has a healthy

curiosity concerning the science whose development has made possible this age of the machine.

Mrs. Logsdon has produced a very readable little book, and by avoiding technicalities of manipulation and taking every opportunity to bring in historical notes—one might almost say historical gossip—she has been able to give the reader a pleasant account of the principles of arithmetic, algebra, pure and analytical geometry, trigonometry and the calculus. Professor G. A. Bliss has contributed a short but striking chapter on " Mathematical interpretations of geometrical and physical phenomena ", which brings out clearly the difference between the " approximative " character of a mathematical theory of some group of physical phenomena and the " axiomatic " character of mathematical theory as a thing in itself.

It is to be regretted that the book was not more carefully written. Some of the historical comments are inaccurate or misleading : for instance, it was the work of the great French mathematicians of the eighteenth century which the Analytical Society admired and which caused the acceptance of Leibnitz' notation. Some of the semi-philosophical comments are puzzling : of the equation

$$(x-x_1)^2+(y-y_1)^2=r^2$$

we are told that " such a general formula would be impossible without the use of analytical geometry " ; this may be true, but does it mean anything ?

Finally, I am prepared to believe that the best method of dealing with limits at this stage is to conceal the difficulties ; but to be told on p. 113 that

$$\lim_{t\to 1}\frac{3t^2-3t}{t^2-1}$$

is indeterminate, and on p. 114 that

$$\lim_{\Delta x\to 0}\frac{2\,.\,\Delta x+(\Delta x)^2}{\Delta x}$$

is found by substituting $\Delta x=0$ after dividing by Δx, is to make me wonder what the appropriate American comment would be. T. A. A. B.

Elementary Arithmetic for the Junior School. I. Teacher's book. By A. H. Russell. Pp. 76. 1s. 6d. 1935. (Harrap)

This book deals with the teaching of addition, subtraction, multiplication and short division. It seems very suitable for its purpose, except that Exercise 103 seems misplaced. It surely belongs to a future Book Two or Three !

E. M. R.

Elementary Arithmetic for the Junior School. II. By A. H. Russell. Pupils' book. Pp. 67. 9d. Teacher's book. Pp. 95. 1s. 6d. 1936. (Harrap)

This second book deals with long division, and introduces simple money calculations—addition, subtraction, multiplication and short division. Sundry useful little " dodges " are given, such as " to multiply by 25, multiply by 100 and divide by 4 ". The exercises are numerous and varied, and there is a useful set of test papers at the end of the book. E. M. R.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW